

Counterexamples

Topics: logic, deduction, mathematical argument, communication

Materials: None

Common Core: Variable, but especially MP3

Prove the teacher wrong. Rigorously.

Why We Love Counterexamples

Every kid loves to prove the teacher wrong. With Counterexamples, they get to do this in a productive way, and learn appropriate mathematical skepticism and communication skills at the same time.

It is possible to play Counterexamples with kids as young as kindergarteners as a kind of reverse “I Spy” (“I claim are no squares in this classroom. Who can find a counterexample?”). What’s great, though, is that you can transition to substantial math concepts, and address common misconceptions. Counterexamples is a perfect way to disprove claims like “doubling a number always makes it larger” (not true for negative number or 0) or sorting out why every square is a rectangle, but not every rectangle is a square. For older kids, you can even go into much deeper topics, like: “every point on the number line is a rational number.”

The language of counterexamples is crucial to distinguish true and false claims in mathematics; this game makes it natural, fun, and plants the skills to be used later. Counterexamples is also a great way to practice constructing viable arguments and critiquing the reasoning of others.

The Launch

Counterexamples is a fun, quick way to highlight how to disprove conjectures by finding a counterexample. The leader (usually the teacher, though it can be a student) makes a false statement that can be proven false with a counterexample. The group tries to think of a counterexample that proves it false.

The best statements usually have the form “All _____s are _____” or “No _____s are _____.” You can also play around with statements like “If it has _____, then it can _____.” For instance:

It’s often best to start with non-mathematical examples.

- All birds can fly. (Counterexample: penguins)
- No books have pictures in them.
- All books have pictures in them.
- If something produces light, then it is a light bulb.
- If something has stripes, then it is a zebra.

Once students have the hang of it, make the examples more mathematical.

- Doubling any number makes it bigger. (Counterexample: -1 doubled is -2 , which is smaller.)
- Multiplying two numbers never gives the same answer as adding them. (Counterexample: $2 + 2 = 2 \times 2$. Or $3 + 1.5 = 3 \times 1.5$.)
- Fractions are always between 0 and 1.
- No square has a perimeter equal to its area. (Note: “equal” isn’t quite right, since units are different. Counterexample: a 4 by 4 square.)

Example

Teacher: I claim all animals have four legs. Who can think of a counterexample?

Student 1: A person!

Student 2: A spider.

Student 3: A fish.

Teacher: Why is a person a counterexample?

Student 4: Because it has two legs.

Teacher: Right. I said every animal has four legs, but a human being is an animal with just two legs. So I must have been wrong. What about this one: everything with four legs is an animal.

Student 5: A spider.

Teacher: A spider is an animal with eight legs, so it proves that not every animal has four legs. But I claimed that if you have something with four legs, it must be an animal. To prove me wrong, you have to give me something with four legs that isn’t an animal.

Student 5: Like a table?

Teacher: Who can tell me if a table is a counterexample to my claim?

And so on.

Tips for the Classroom

1. It’s good to make up false conjectures that are right for your students. But start simple.
2. For young students, use silly claims, e.g., “The only one who likes cookies is cookie monster.”
3. Kids can think of their own false claims, but sometimes these aren’t the right kind, and they often have to be vetted.
4. Once you introduce the language of counterexamples, look for places to use it in the rest of your math discussions.
5. You can also use Counterexamples to motivate a normal math question. Instead of saying “draw a triangle with the same area as this square,” you can say, “I claim there is no triangle with the same area as this square.” If students know to look for counterexamples, this will set them to work trying to disprove the claim right away.

References: <http://kuow.org/post/getting-kids-interested-math-without-their-knowing>

Counterexamples in Action: Pattern Blocks

Here are a series of Counterexample challenges that launch an exploration with pattern blocks. These level up in difficulty to the main exploration (challenges 4 & 5).

1. “It's impossible to make a hexagon with pattern blocks that isn't yellow.”
2. “It's impossible to make a hexagon that isn't yellow or red (assuming they made a red hexagon from two trapezoids).”
3. (If they've made hexagons with 1, 2, 3, and 6 blocks) It's impossible to make a hexagon using exactly 4 or 5 blocks.
4. “Ok, it's definitely impossible to make a hexagon that uses MORE than 6 blocks.”
5. You can launch them onto their own with this question: Is it possible to build a hexagon with ANY number of blocks? To start, can you build a hexagon using 7 blocks? 8 blocks? 9 blocks? ... up to 20 blocks?
6. Bonus (harder): “Is is impossible to build a hexagon with 50 blocks/100 blocks.”
7. Bonus: “It is impossible to make a hexagon using only squares”

Along the way, you can draw kids attention to any nonstandard hexagons that get built. That fifth question is the key one for them to work on on their own for a longer time. Ideally, they can build hexagons with larger numbers of blocks, and check off the numbers they're able to get. It would be cool if, as a class, you could get all the numbers from 7 to 20, or even higher.