PUZZLE 1

There are 34 ways to write 8 as a sum of 1s and 2s.

To see why, let's start simple and see if we can get a handle on what's happening.

1: 1 (1 way)
2: 1 + 1, 2 (2 ways)
3: 1 + 1 + 1, 2 + 1, 1 + 2 (3 ways)
4: 1 + 1+ 1 + 1, 1 + 1 + 2, 1 + 2 + 1, 2 + 1 + 1, 2 + 2 (5 ways)

Starting small allows us to look for some labour-saving insight. What if we want to break 5 up into 1s and 2s. The start will be either 5 = 1 + ... or 5 = 2 + ...

But this reduces the problem to a simpler situation! More specifically we have 5 = 1 + [some way to make 4 with 1s and 2s] or 5 = 2 + [some way to make 3 with 1s and 2s]. We know that there are 5 ways to make 4 and 3 ways to make 3, so there must be 5 + 3 = 8 ways to make 5.

5: (8 ways)

And now we can see what's happening. The number of ways to make each subsequent number on the list is the sum of the number of ways to make each of the previous two. So the list will continue

6: (13 ways)
7: (21 ways)
8: (34 ways)

These are Fibonacci numbers! It may be a surprise, but each Fibonacci number is the sum of the previous two, and that's precisely how our partitions work too.

PUZZLE 2

There are 21 ways to write 8 as a sum of three positive integers. Here's one beautiful structure that emerges when we try to write all the ways out, putting each sum that starts with a given number on its own line:

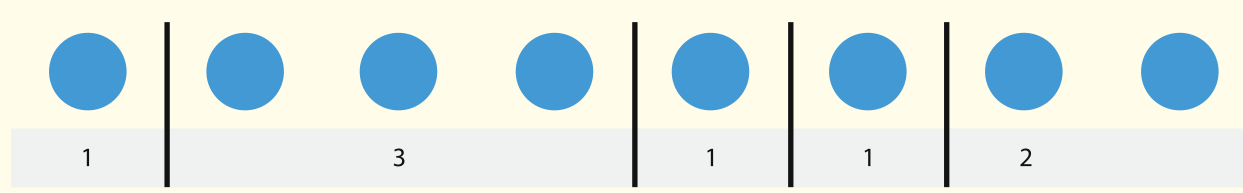
6 + 1 + 1					
5 + 2 + 1	5 + 1 + 2				
4 + 3 + 1	4 + 2 + 2	4 + 1 + 3			
3 + 4 + 1	3 + 3 + 2	3 + 2 + 3	3 + 1 + 4		
2 + 5 + 1	2 + 4 + 2	2 + 3 + 3	2 + 2 + 4	2 + 1 + 5	
1 + 6 + 1	1 + 5 + 2	1 + 4 + 3	1 + 3 + 4	1 + 2 + 5	1 + 1 + 6

It's a triangle! And there are a triangle number of ways to write 8 as a the sum of 3 positive integers: 1 + 2 + 3 + 4 + 5 + 6 = 21 ways in all.

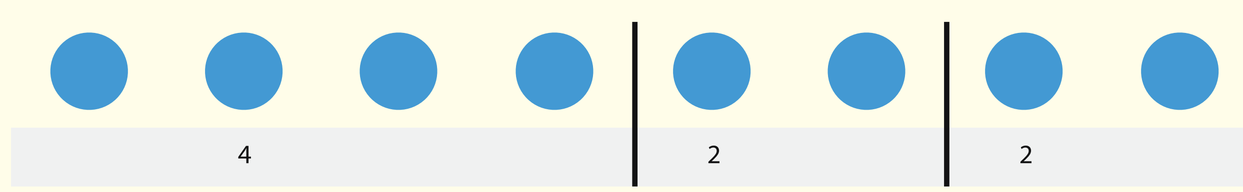
PUZZLE 3

There are 128 ways to write 8 as a sum of positive integers. Let's imagine 8 objects that we'll literally partition into groups.

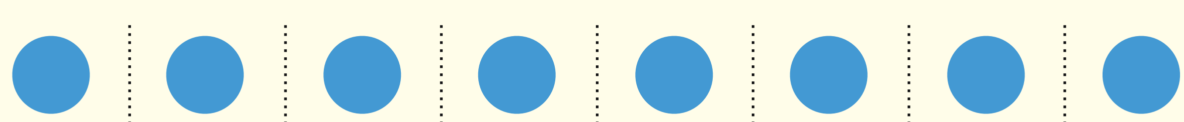
For example, here's 1 + 3 + 1 + 1 + 2



And here's 4 + 2 + 2



It's clear that each partition corresponds to a way of, well, partitioning the dots. So how many ways can we do this? There could be a partition between any two dots, so that gives us 7 places we could draw in a barrier.



In any sum, each partition is either there or it isn't. There are two choices for each, so we double the number of options for our sum when we consider each possible line between dots. That means there must be 2 x 2 x 2 x 2 x 2 x 2 x 2 = 2^7 = 128 ways to write 8 as a sum of positive integers.

Three puzzles, three different approaches. I encourage you to look further: there are many more ways to approach all these puzzles, and much more waiting to be discovered. Happy puzzling!