THE HINDU IN SCHOOL
SUNDAY, FEBRUARY 4, 2018

These puzzles, as with many others like them, can be solved by starting simple, and working your way up. Make a table comprising the first few fair and unfair numbers, and you'll notice something right away.

Fair Numbers	Unfair Numbers
1	3
2	5
4	6
8	7
16	9
32	10
64	11

It sure looks like the fair numbers are precisely the powers of two, doesn't it? If so, that solves all three puzzles. But how can we prove that this is right?

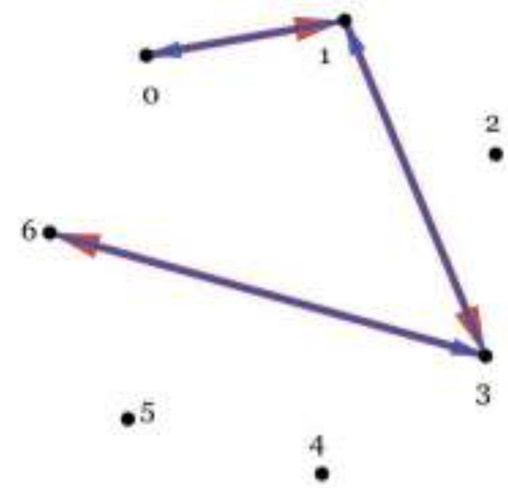
Some part of the key lies in reducing the numbers a bit. If there are 10 people in our circle, then the ball traveling forward 8 is the same as traveling backward 2. In other words, we can think of 8 and -2 as being equivalent to each other.

Welcome to the world of modular arithmetic, where number circles are the norm.

It takes a little getting used to, but instead of the steps forward being 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, we can think of them as 1, 2, 3, 4, 5, -4, -3, -2, -1, 0, and then repeating. (This is for ten people only. For different numbers, the equivalences will change.)

This insight allows us to see that odd numbers will never work. Consider 7. With 7 people in a circle, the steps would be: 1, 2, 3, -3, -2, -1, 0, and then they would repeat.

We barely even need to draw this out: it's clear that the step of -3 will undo the step of 3, and we'll begin retracing our steps to the beginning, and then begin again.

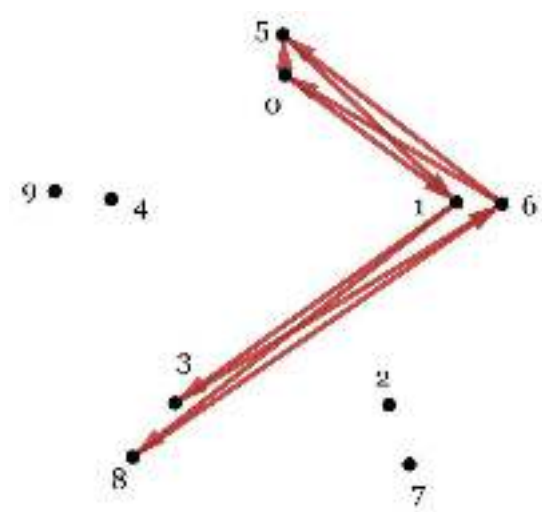


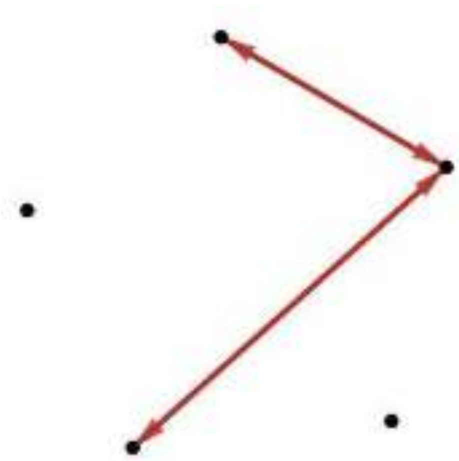
This same undoing effect will happen with every odd number.

So all odd numbers (aside from 1), must be unfair.

What about even numbers, then? It seems like there's a lot of possibilities to check, but one thing to note is that the steps forward here will always have a "middle." Remember the steps for 10: 1, 2, 3, 4, 5, -4, -3, -2, -1, 0. Add them together and they add up to 5, since everything else cancels. That means we'll end up after ten throws with the ball at the opposite side of the circle.

It would be nice if it had come back, and here's where we can try out a very tricky idea: what if we had everyone stand with the person who was across from them, in a double circle? In this case, one circle of 10 becomes a double circle of 5. Keeping very close track of the path of the ball, we see that the movement in the double circle of 5 matches nicely on to the movement of a ball in a regular, single circle of 5.





There are some details I'm leaving out, but this is actually a general pattern at play. Take your single circle and wind it around twice, and the path of the ball will map right on to the path it takes in the single circle. So if you double-wound your circle and got an odd number of spots, the number you started with was unfair.

If you get an even number of spots, guess what: you double-wind your double-wound circle again. Keep going, and if you ever get an odd number other than 1, your starting number was unfair. If you do get 1, then you must have started with a power of 2!

It takes a bit more work to show that all the powers of 2 are fair. We can use a mathematical argument known as induction to do so. But for now, let's enjoy the argument we have built already, and digest the details.

Here's a **RESEARCH QUESTION** to consider: what if instead of throwing the ball forward 1, 2, 3, 4,..., you picked a different sequence, like 1, 3, 5, 7, 9,... , or 1, 4, 7, 10, 13,...? What numbers will be fair or unfair for each of these sequences?

You can write to Dan Finkel (dan@mathforlove.com) with your responses to the Research Question [subject: 11Play].