PUZZLE 1: THE POWER OF 37

No matter what digit you pick, the answer will always be 37. That's the power of 37! But why does 37 have this power?

The key observation is that no matter what digit you pick, repeating that digit three times in a row is the same as multiplying it 111 times.

And 111 is divisible by 37!

$$111 = 3 \times 37$$

Suppose we choose 8 as our digit. Our chain of equations looks like this now:

$$888 = 8 \times 111 = 8 \times 3 \times 37 = (8 + 8 + 8) \times 37$$

Or, dividing both sides by (8 + 8 + 8), we get

$$888 \div (8 + 8 + 8) = 37$$

Nothing about 8 was important in this equation, and indeed, the 8 could have been any other digit. The answer would always be 37.

PUZZLE 2: THE 7, 11, 13 MYSTERY

This mystery has a similar solution to the last, though the details are a bit more involved.

The observation in this case is that writing a three digit number (745, say) twice in a row is the same as multiplying it times 1001.

It turns out that 1001 is the product of 7, 11 and 13.

$$1001 = 7 \times 11 \times 13$$

So

$$7{,}45,745 = 745 \times 1001 = 745 \times 7 \times 11 \times 13$$

So when we divide by 7, 11, and 13, we cancel out the numbers we multiplied by when we wrote the number twice in a row, and are left with the original number. So this trick always works as well.

$$\frac{745 \times 7 \times 11 \times 13}{7 \times 11 \times 13} = 745$$

PUZZLE 3: NUMBERS THAT BREAK CALCULATORS

This puzzle has a different flavour. The key insight is to start simple and build on from there. We will do this by considering how far away each tower is from being a multiple of 11.

10^1 = 1 less than a multiple of 11.
$10^2 = 100$ = 1 more than a multiple of 11.
10^3 = 1 less than a multiple of 11.
10^4 = 1 more than a multiple of 11.

This pattern will continue. (Do you see why?)

So we just need to consider whether 10 is raised to an even or odd power.

Since 5 to any power is odd, 10 will be raised to an odd power, and hence the entire tower will be 1 less than a multiple of 11.

This implies that the second part of the sum will be 1 less than a multiple of 11.

Following similar reasoning, and noting that increasing each power of five is the same as multiplying the last number by 5, we see that:

5^1 = 5 more than a multiple of 11.
5^2 = (multiple of 11) + 3 more than a multiple of 11.
5^3 = (multiple of 11) + 15 = (a bigger multiple of 11) + 4.
5^4 = (multiple of 11) + 20 = (a bigger multiple of 11) + 9
5^5 = (multiple of 11) + 45 = (a bigger multiple of 11) + 1.

So 5^5 is 1 more than a multiple of 11.

The pattern will continue from here, and every fifth power of 5 will be one more than a multiple of 11.

Since 5 is raised to some enormous power of 10 in the tower, the entire tower will be a fifth power of 5.

Therefore, the first part of the sum will be 1 more than a multiple of 11.

So from our original towers, we can see that one is one more than a multiple of 11, and the other is one less than a multiple of 11. Add them together, and you get a multiple of 11.
So the sum is divisible by 11.

I love this problem. To solve it, you need to take seemingly unworkable numbers and find a way to work with them.

The technique of looking at remainders, or how far off the number in question is from a certain number is formalised in a type of math called modular arithmetic. It's a truly beautiful subject, and incredibly powerful, particularly when it comes to handling unfathomably large numbers.