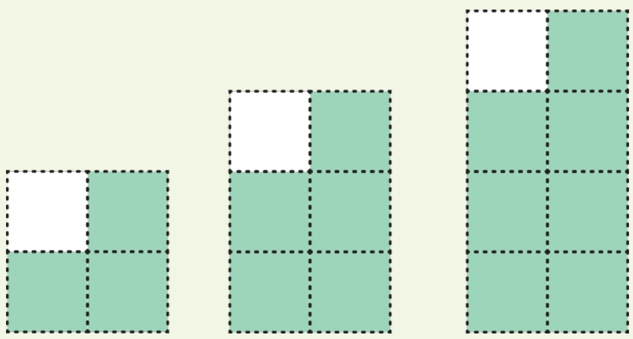
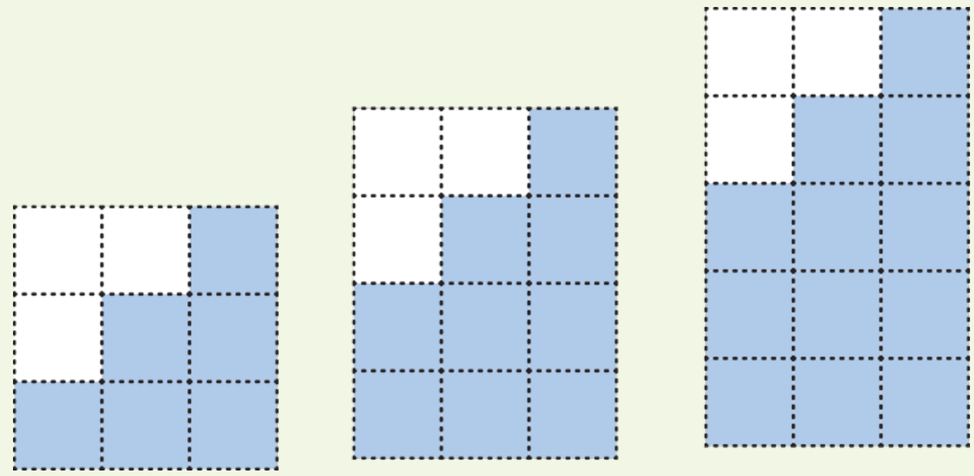
Did you draw pictures of staircases? If so, they might have helped you!

PUZZLE 1

The 2-step numbers are precisely positive odd numbers 3 and above. To see this, consider the smallest 2-step number you can make, and let it grow as in the figure below.



This series starts with 3, and grows by 2 each term. That generates all the odd numbers. On the other hand, every 2-step number is an even number plus an odd number, which means every 2-step number must be odd.



Similarly, every 3-step number is a multiple of 3 larger than 6. You can see this by starting from the smallest staircase and adding 3 each term.

Notice that you can also take a square from the big step and move it to the small step to make a rectangle every time!

The logic continues to work.
2-step numbers = (multiples of 2) + 3
3-step numbers = (multiples of 3) + 6
4-step numbers = (multiples of 4) + 10
5-step numbers = (multiples of 5) + 15
6-step numbers = (multiples of 6) + 21
And if you know algebra, you can write in general
k-step numbers = (multiples of k) + (1+2+3+...+k)

You can use algebra to change these descriptions into various forms if you'd like.

For example, since 15 is itself a multiple of 5, 5-step numbers are just multiples of 5 that are equal or greater than 15. This helps us solve Puzzle 2.

PUZZLE 2

We want numbers that are 2-step, 3-step, and 5-step all at once.

Look at our solutions from Puzzle 1, and we know that we are looking for numbers that are:

1) Odd
2) Multiples of 3
3) Multiples of 5

The first is 15, which we already knew.

Go forward 30 (2 x 3 x 5 = 30) numbers and we'll have 45, which should also be 2-step, 3-step, and 5-step.

And it is, because
45 = 7+8+9+10+11 = 14+15+16 = 22+23

Continue adding 30 and you'll continue to get numbers that are 2-step, 3-step, and 5-step: 15, 45, 75, 105, 135, etc.

PUZZLE 3

This is the extraordinary one, in my opinion. It turns out that the numbers that cannot be written as step-numbers are precisely powers of 2, so there are 10 less than 1000:

1, 2, 4, 8, 16, 32, 64, 128, 256, 512.

Why this beautiful pattern? Let's look at what numbers are step numbers: all odd numbers, all multiples of 3 (starting from 6), multiples of 5 (starting from 15), multiples of 7 (starting from 28), and so on.

If you strike every number from the list of positive integers that are odd or divisible by an odd number, all that remains is powers of 2. (This is a good fact to convince yourself of if it isn't clear.)

It takes a little more work to see that no power of 2 can be a 4-step, 6-step, 8-step, etc. number, but it follows from the descriptions of those numbers, and I'll leave it as an exercise for the reader.

I've always loved the fact that **powers of 2 have this magical property**, that they are the only numbers that cannot be written as step numbers. If you want to play further, you can explore **what numbers cannot be written as step numbers with 3 or more steps**. In this case, **our answer expands from powers of 2 to also include prime numbers**.

This is a good thing to prove on your own, and a great example of how mathematical structures can put surprising answers. There are very few places I'd expect to see powers of two and prime numbers lumped together, but this is one of them!