All these puzzles can be solved by appealing to parity, or evenness/oddness. We saw parity come up before when we explored the Billiard Ball problem, and its a surprisingly useful tool for solving all kinds of puzzles. Let's see how it helps us here.

PUZZLE 1

You may have noticed that you can get any odd number between -15 and 15, but no even numbers. Why? We can prove that even numbers will never come up by paying attention only to the parity of each of our numbers:

__1__2__3__4__5 = ?

↓

__odd__even__odd__even__odd = ?

If you've ever played around with adding even and odd numbers, you know that adding or subtracting an odd number changes the parity of your sum/difference, while adding or subtracting an even number keeps the parity the same. Since there are three odd numbers, the final total must be odd. That's all there is to it!

Just play around with the numbers and you'll find it isn't too hard to get all the odd numbers from -15 to 15. For example:

1 + 2 - 3 - 4 + 5 = 1

1 - 2 + 3 - 4 + 5 = 3

And so on.

BONUS PUZZLE: -1 + ((2 + 3) x 4 x 5) = 99.

PUZZLE 2

The rook must start on a white square to finish the tour. Again, parity helps us understand why this must be so. The rook's tour alternates between white and black squares. Speak them out as the rook travels and it would go "black white black white..." or "white black white black..." until all 25 squares were covered. But 25 is an odd number, which means the ending square would be the same as the starting square:

"black white black white... black" or "white black white black... white"

Stack up those black or white squares and we see that the tour must touch 13 of whatever colour the rook starts on, and 12 of the other colour. Now look at the board: there are 13 white squares and 12 black ones. The rook couldn't start on a black square and finish the tour. It wouldn't add up!

This is a different use of parity, but again, evenness and oddness is key. If the board were 4 by 4, the rook could have started on any squares and still completed the tour.

PUZZLE 3

The prisoners can save everyone except the person at the back of the line, who has a 50/50 chance.

Once again, it all comes down to parity. Here's the strategy:

Imagine each black hat is worth 1, and each white hat is worth 0. Then add up the value of all the hats.

(This is another angle on odd vs. even: adding 1 changes the sum from odd to even and from even to odd; 0 leaves it the same.)

The prisoner in the back should look ahead, find this sum, and say "black" if the sum is odd and "white" if the sum is even.

If he's lucky, he'll survive. Hopefully he'll be lucky.

Everyone else will be okay, though! Consider this:

Let's say the prisoner at the back said "black."

The prisoner second from the back now knows that that prisoner saw an odd number of black hats.

He looks ahead and sees an even number of black hats.

That means he must be wearing a black hat! So he says "black."

The next prisoner hears this, and knows that there must now be an even number of black hats.

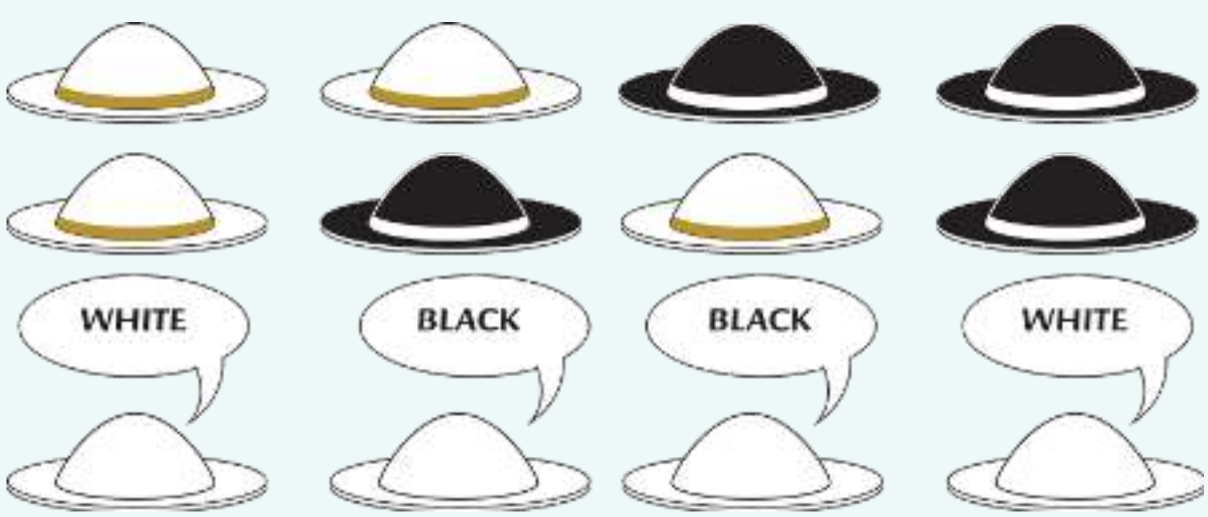
If he sees an even number of black hats ahead of him, he'll say "white."

If not, he'll say "black."

The parity information keeps getting passed forward, and each prisoner just needs to track whether the number of black hats ahead is even or odd, based on the information from the prisoners behind.

A strong recommendation to help make sense of this one: try it with just three prisoners (shown below). What happens?

What about with four prisoners?



The four possible cases in this puzzle when three prisoners are involved are shown in this illustration.

In the first case, the last person sees two white hats. Since the sum is zero (0+0), he shouts out "white". Knowing this and seeing a white hat in front of him, the second person will realise that both of them are wearing white hats. So he will also shout "white". On hearing this the person standing in front will know his hat is also white and hence he will also shout out "white".

In the second case, the last person will shout out "black" (0+1). The second last person will see a white hat in front of him and realise his hat colour and hence shout "black". Once the person in front hears this, he'll know his hat colour and shout "white".

Similarly, other two cases can also be explained.

For all these puzzles, parity helps to give us what mathematicians call an invariant. As the name suggests, this is a way of constructing some relationship that doesn't change, even as everything else is in chaos. Invariants are great! They are a kind of north star of mathematics, and when we find them we can use them to navigate even in strange and uncertain waters. Parity is powerful precisely because it gives us invariants so reliably. There are lots of other kinds of invariants as well. I'll discuss them in the future. Till then, happy puzzling!