A MATHEMATICIAN AT PLAY

Playing with conjunctions and dissections

One of the delightful aspects of shapes is how they can be cut up and recombined in surprising ways. Daniel Finkel does just that as he poses three puzzles where you will have to cut the shapes and combine them in different ways to get to your answers.

Many years ago, a student I was working with asked me if you could take any collection of congruent squares and cut them apart, rearrange the pieces, and form a single square. We worked for many weeks to convince ourselves that it was indeed possible to form a single square from any collection of smaller squares.

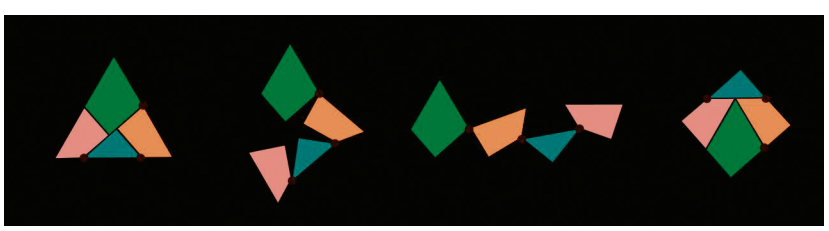
In fact, this was a problem that had been considered by mathematicians in the early 19th century. The Wallace-Bolyai-Gerwien Theorem, proved independently by the three mathematicians it is named for, states that any polygon can be dissected and recombined to form any other polygon of the same area. David Hilbert posed the question of whether the analogous question was true for 3-dimensional polyhedra as well. (The answer, it turned out, is no.)

If you're up for a challenge, try to prove that any rectangle can be dissected to form a square. That's the key step of the proof of Wallace-Bolyai-Gerwien, which goes roughly like this:

1) Any polygon can be cut into triangles
2) Any triangle can be cut into a rectangle
3) Any collection of rectangles can be cut into a single rectangle
4) Any rectangle can be cut into a square.

Do these steps forward and backward, and you've cut one polygon into another. You just need to prove the steps will always work.

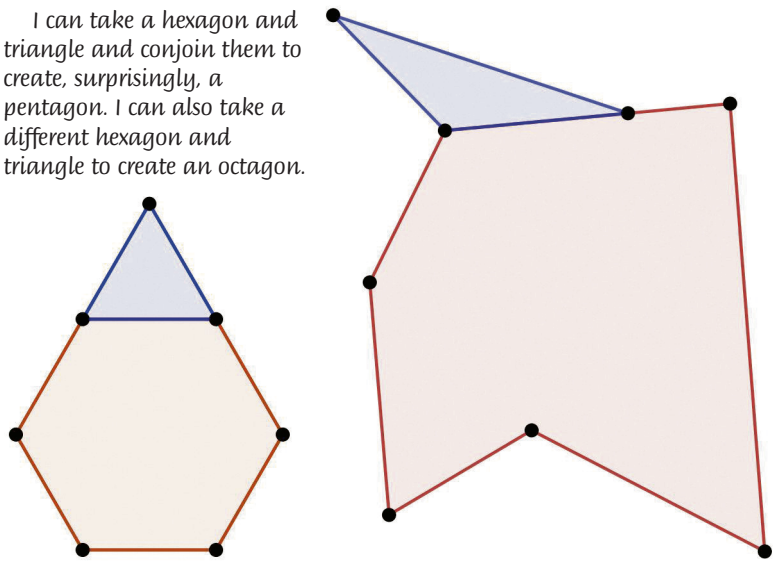
While the theorem handles the general situation, amateur mathematicians and expert puzzlers worked to create dissections that, with a minimum number of cuts, would actually transform one shape into another. I first saw the dissection of a square into an equilateral triangle in one of Dudeney's puzzle books; it take just four pieces.



Today, I'd like to share three puzzles that involve dissecting or conjoining shapes to make new ones. These are best done by drawing or, even better, cutting out shapes and moving them around.

PUZZLE 1

I can take a hexagon and triangle and conjoin them to create, surprisingly, a pentagon. I can also take a different hexagon and triangle to create an octagon.



Find ways to conjoin one hexagon with one triangle to make each of the following: a triangle, a quadrilateral, a pentagon, a hexagon, a heptagon (a 7-sided polygon), and a nonagon (a 9-sided polygon).

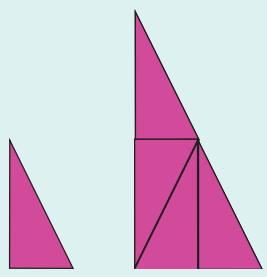
PUZZLE 2

It is possible to dissect an obtuse triangle into acute triangles?

What is the minimum number of acute triangles it takes to perform this dissection? (No obtuse or right triangles are allowed when the dissection is complete!)

PUZZLES 3

It is possible to take any four congruent triangles and conjoin them to form a similar triangle with 4 times the area. For example, it's possible to take four copies of this 1 by 2 right triangle and make a larger version of the same triangle.



a) Can you figure out how to build a larger version of the same triangle with 5 copies of the triangle?

b) Can you take a different familiar triangle and build a larger version with just 3 conjoined copies?

Dan Finkel is the founder of Math for Love, an organisation devoted to transforming how math is taught and learned. He is the creator of mathematical puzzles, curriculum, and games, including the best-selling *Prime Climb* and *Tiny Polka Dot*.