Nine Points on Rich Math Tasks

by Daniel Finkel

When it comes to possibilities for math class, I use many of the following words synonymously: puzzles, problems, tasks, explorations, questions, activities. Each has a different shade of meaning, and probably many of us use them differently. What is important though, is that students have a pathway to a rich learning experience that challenges, engages, and deepens their learning.

My goal in this article is to inquire into what makes mathematical tasks this sort of rich learning experience for students in the classroom. I advocate for encounters with beautiful, transformative mathematics that are accessible and appealing to students. Engagement in this kind of experience is precisely what I mean by richness.

The critical element of richness is that it describes what our students experience, rather than some inherent quality of the activity. I have had conversations about long division that were incredibly rich, counter to my expectations. The opposite is all too common as well: the task you think will be rich doesn't work in the classroom, or even more confusingly, is amazing in one class but doesn't work in another.

Let's consider a problem that I believe has the capacity to be quite rich.

The problem: How many different ways can you fill in the blanks with positive integers to make this equation true?

8 = ________ + _________ + _________

Point 1. The richness of a task depends on how your students experience it.

One student's rich task is another student's tedious exercise, and another's intractable problem. If a student doesn't care about it, isn't grabbed by it, finds it too hard, too easy, too pointless, then any richness that could potentially emerge from the problem never does. We need to be aware of the potential pathways to and beyond the problem.

It may be easy to come up with one solution to this problem, but for fifth graders, for example, it is probably difficult to come up with all the solutions so that students might check out, or require more prompting to really engage. The point is:

Point 2. The task is only rich if students are motivated to engage in it.

So, before we pose a question, we need to think about why it's actually interesting. I'm a big fan of the inherent interest in mathematical problems, and from what I've seen, students in fifth grade and up are pretty interested in abstract counting problems, these "what are all the ways" type of problems.

However, we can motivate it with a simple warmup problem, around which we could lead a class discussion. Let's imagine a dialog.

Warmup: Fill in the blanks to make this equation true.

8 = ________ + _________

Student 1: 5 + 3

Student 2: 7 + 1

Student 3: 8 + 0

Teacher: How many ways do you think there are to fill in the blanks?

Student 4: Infinity. Because I could do 7.5 + 0.5, or 7.6 + 0.4. As many different decimals as I want.

Teacher: That's a good point. Ok, I'm going to change the problem a little so the answer isn't infinity anymore.

Updated Warmup: How many ways can you fill in the blanks to make this equation true, using whole numbers?

Student 5: Can I use negative numbers?

Teacher: How would you use them?

Student 5: Well, -1 + 9, -2 + 10, -3 + 11.

Teacher: What do the rest of you think? Do these work?

Student 6: They all equal 8! There will still be infinitely many ways to solve it!

Teacher: I agree. I'm going to alter the problem again.

Updated Warmup: How many ways can you fill in the blanks to make this equation true, using positive whole numbers?

Student 3: But that won't let me use 8 + 0.

Teacher: True. Let's add this challenge for later. [Teacher writes on the corner on the board: "How will allowing 0 change the answers?"] For now everyone take a minute and figure out how many possibilities there are altogether.

[Students work in pairs or trios for a minute or so.]

Student 7: There are four different answers.

Student 8: I got seven.

Teacher: That's strange. Could we list them all?

[Students 7 and 8 write their lists down on the board.]

Student 7: You shouldn't include 5 + 3. It's the same as 3 + 5.

Student 8: I think they're different.

Teacher: This is a decision we should talk about.

Teacher: Something you both did that I like a lot was to organize your lists. I can see all the possibilities very clearly.

Teacher: Okay, two more warmups, and you can choose which one you want to try, or do both. How many ways are there to fill in these blanks to make each of the equations true, with all the same rules as before?

6 = ________ + ________

40 = ________ + ________

Let's take a minute and look at what's happening with this problem so far. The teacher is starting from a place that should be easy for everyone, and everyone is getting some initial successes as they get oriented on the kind of thinking this problem requires. The teacher is also choosing to open up the discussion to include what rules and conventions will make this problem the most interesting. Allowing non-integers and negative numbers will immediately lead to "infinity" as an answer; that's less interesting (at this moment, for these students), so we hone in on what will make the problem interesting. Other issues, like whether 5 + 3 should be counted as distinct from 3 + 5, are something the teacher may know by virtue of playing around with the problem before.

Even though we haven't gotten to our main problem yet, we have the students engaged and thinking, having initial successes, and clear on the technical details. We've essentially created a runway leading to the problem, that will allow them to fly once they really get it.

Point 3. Rich tasks should either be quickly accessible, or, students should have the tools to make them accessible.

The main thing is to get students thinking and engaged as quickly as possible. You can do this by posing the question right away, posing easier questions to build up to it, posing harder questions, or involving students in a conversation.

I often imagine our goal as helping students to change the step size they encounter when they work on a problem. In other words, when they reach a problem that feels like an impossible step up, I want them to have the tools to break it into a staircase (see Figure 1). In this case, the first step isn't too hard. But if it were, I'd want them to try to get just a single answer to make 8 = ___ + ___. Or change the 8 to a 4.

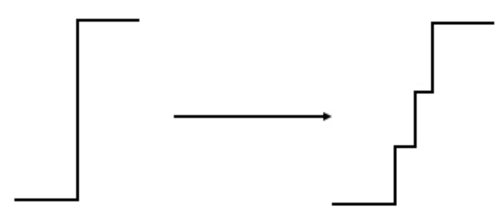

Figure 1 An impossible step gets broken into manageable steps.

On the other hand, if the step feels too small, I want them to think about how they could make it harder. How could we answer the question if the 8 were a 40? Is there a way to quickly solve the problem for any number?

Point 4. Rich tasks balance success and challenge.

This illustrate a balancing act: problems that are too hard or too easy won't hold students' attention for long. (They tend to call both types of problems "boring.") Perfectly crafting the experience for all students isn't realistic, but with rich problems, we don't have to: they are adjustable by the students, and once they get engaged, they tend to find ways to strike this balance on their own.

Point 5. Rich tasks teach habits of mind and content simultaneously.

Of course, we can help them if we teach them habits of mind that allow them to take control of their experiences in tackling the problems. If students feel like they own the problem, then they can take liberties like changing the problems to make them simpler. This maneuver is actually a profoundly useful problem-solving technique: start simple, and slowly build the complexity back in. If you add in a table, chart, graph, or other organizational tool, you've got a recipe to overcome a lot of what's going to be presented in a mathematics class.

Returning to the classroom example, we might expect students to notice that there are five ways to make 6 from two addends (1+5, 2+4, 3+3, 4+2, 5+1), and 39 ways to make forty. It might be worth a quick discussion of the general structure here (are there always n−1 ways to make n from two addends? Why?). Or, we might choose to go right into the main problem.

The problem: How many different ways can you fill in the blanks with positive integers to make this equation true?

8 = ________ + _________ + _________

All the habits of mind—i.e., starting simple, organizing your work—that came up before will be useful here too. But at this point, I'd expect most students to be ready for this much harder problem. The rules are clear, they've already had a success, they are ready for a greater challenge, and they (rightly) anticipate that there's something cool to find out.

And now, we let them work. It depends on how much stamina they have built up working on problems of this kind, but I would expect that students could work on their own for 20 minutes to 40 minutes, with the teacher talking to pairs and trios to further their thinking.

Point 6. You need to try out rich tasks on your own before you give them to your students.

Seriously. Take a minute and solve this problem. How do you approach it? What's your answer, and more importantly, what is your pathway towards your answer? We have to do the math we ask students to do, or we won't have a personal experience of what the various pathways through a problem are, and how they feel.

Point 7. There is more than one way to solve a rich task.

I recently posed this problem in a fifth grade class, and there were at least three ways students approached it, one of which I had never seen before.

Method 1: Choose the highest number you can first, and then systematically make a pattern working down.

6 + 1 + 1	5 + 2 + 1	4 + 3 + 1	3 + 4 + 1	2 + 5 + 1	1 + 6 + 1
	5 + 1 + 2	4 + 2 + 2	3 + 3 + 2	2 + 4 + 2	1 + 5 + 2
		4 + 1 + 3	3 + 2 + 3	2 + 3 + 3	1 + 4 + 3
			3 + 1 + 4	2 + 2 + 4	1 + 3 + 4
				2 + 1 + 5	1 + 2 + 5
					1 + 1 + 6

There were variations on this too, like starting with the lowest number first. Either way, it yields a beautiful structure: a kind of triangle, organized by columns that each start with different numbers. We can count the number of ways to solve the problem now: 1 + 2 + 3 + 4 + 5 + 6 = 21 solutions. Moreover, this method feels like a generalization! For the students who solved it, we might ask them to replace the 8 by a 12.

Method 2: Start small, and work your way up.
Students using this method found all the ways to add three positive integers to make 3 (there's just one way: 1 + 1 + 1), and then 4, and on up. They ended up with a table:

Sum	3	4	5	6	7	8
Number of ways to make the sum with 3 addends	1	3	6	10	15	21

This is pretty nice! There's a pattern in these numbers (lots, in fact), and students could use the pattern that emerges to make predictions about much larger numbers.

Method 3: Account for rearrangements.
This one surprised me. Students considered 8 as the sum of 6, 1, and 1. How many sums can we get from those addends? Three: 6 + 1 + 1, 1 + 6 + 1, and 1 + 1 + 6. What about 5, 2 and 1? In this case, there will be 6 possible rearrangements. It's a subtler argument, and requires a bit more work, but it's quite a beautiful approach.

There are many other approaches to this problem as well.

Point 8. Save time to wrap up.

There were a lot of things to share, and it's good for students to have some time to try to explain what they figured out, and also to be aware of what we still don't know about a problem. Do we have a way to solve the problem for any sum? How many ways could we solve 25 = __ + __ + __ using positive integers in the blank spaces? Why do the different approaches from before all give a right answer, and what does it tell us that they do? In method 3, it seems like every rearrangement is the sum of 3s or 6s, so it seems like the total should be divisible by 3; but in method 2, we saw that not every rearrangement was divisible by 3. Can anyone make sense of that? What if there were four blanks in our equation instead of 3?

The wrap up isn't about knowing everything, but it's important to know what you know, and what you don't. I always like my students to leave with something to reflect on.

For me, introducing richness into mathematics is the raison d'être of mathematics class: to have rich experiences thinking about mathematics. But rich tasks don't have to go perfectly, and they won't. All you really need to get started is an idea for what a pathway through a problem might look like; if it was fun for you to think about, maybe it will be fun for your students too.

Point 9. Go in with an idea for the rich task, plus an easy version, plus an extension.

In this example, I have three levels in my head as class starts, involving counting all the ways to make an equation true by putting positive integers in the blanks.

The main problem: 8 = __ + __ + __

The easier version: 8 = __ + __

The extensions: 14 = __ + __ + __
OR
8 = __ + __ + __ + __

If I focus on whether or not my students are having a rich experience, I can make sure they have enough success to take on the challenge, have enough challenges to be engaged, and have the right amount of engagement for them to discover the beauty and richness of mathematics.

For more ideas about rich tasks, visit Dan's blog https://mathforlove.com/blog/, or follow him on Twitter @MathForLove.