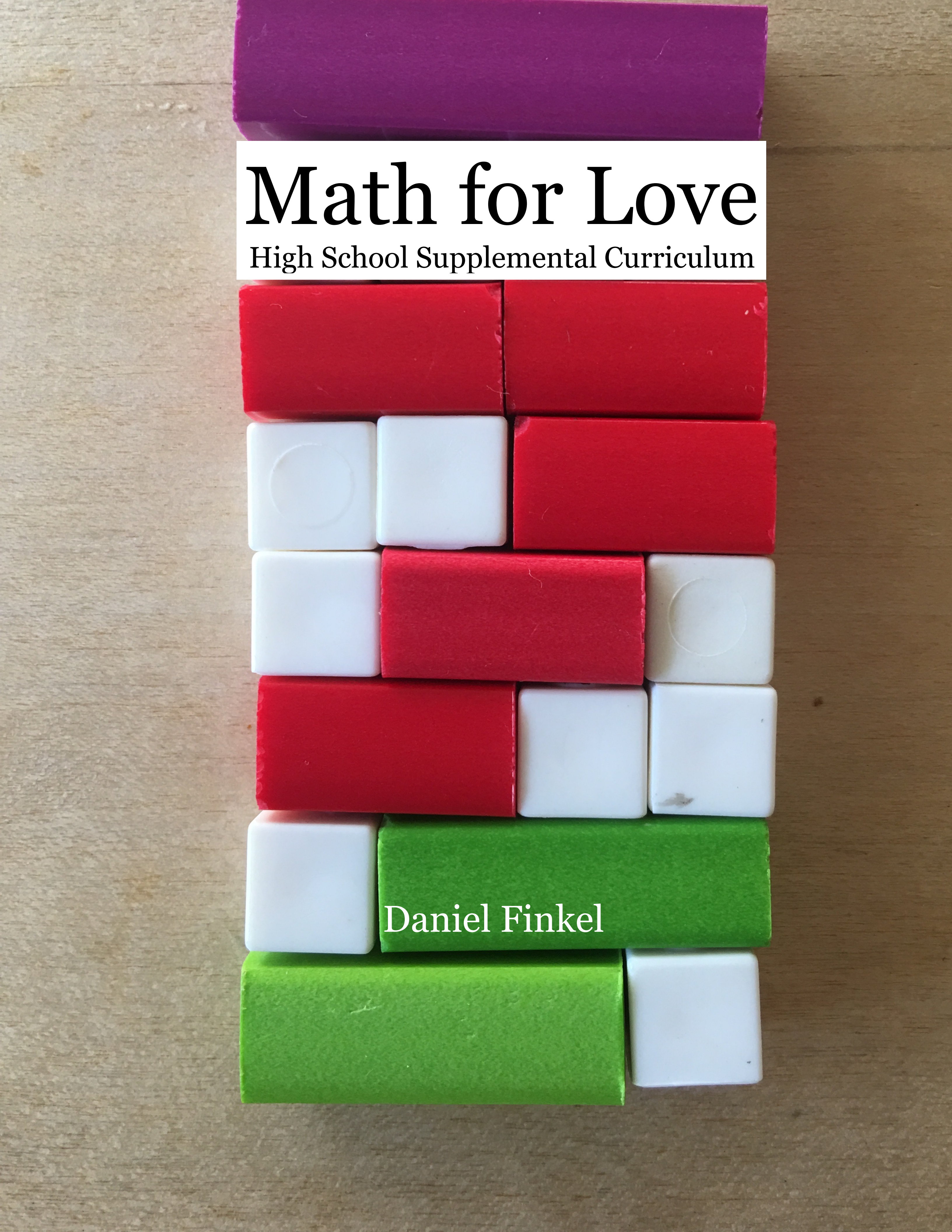
Math for Love

High School Supplemental Curriculum



Daniel Finkel

Introduction

You have in your hands an early version of a supplemental curriculum for high schoolers. I hope you and your students get a lot out of it!

This curriculum uses a series of openers & rich tasks to engage students in problem solving. This could be used a standalone summer curriculum, or as inspirational problems to be incorporated into a class during the school year. Each rich task could be used for just 1 day, or could be expanded into a multi-day experience. The openers are meant to be 10 minute class starters, and are key to making sure students get regular exposure to the thinking and symbols of algebra.

What aligns these problems is their potential connection to algebraic description. For most of lessons, a complete solution will involve writing an algebraic equation to describe the situation we're exploring. By the end of this intervention, students should be more confident and capable in writing algebraic equations to describe potentially complex behaviors and patterns.

Throughout this curriculum, plan to start every day with an *opener*, which will provide a chance to practice writing algebraic descriptions of visual patterns explicitly. Plan to close with an exit ticket.

Day Plan

Opener - 5 - 15 minutes
Rich Task - 40 - 60 minutes
Wrap up/Exit Ticket - 5 - 15 minutes

Use time wisely! Every minute counts for students in danger of falling behind in math.

If time permits, play math games like Prime Climb, 24, Horseshoes, Hex, and Pig in the final minutes of class.

If you have access to computers, students can play **Mobiles** (solveme.edc.org) or **Sumaze** (integralmaths.org/games/sumaze).

Openers

Every day starts with an opener. Usually these are *visual patterns*[1], where the goal is always the same.

1. Describe the 29th (or *n*th) number in the blue sequence.
2. Describe the 29th (or *n*th) in the red sequence.
3. Describe the 29th (or *n*th) in the total (blue + red) sequence.

Students should become more and more capable in describing the sequences algebraically as they get more practice.Rich Tasks
For video support on using rich tasks in the classroom, refer to mathforlove.com/pd.

Anticipate launching students into working on tasks within 5 - 10 minutes. Some tasks also include video launches or slides. Students themselves should make most of the breakthroughs and connections while working on the task. The teacher should be active too, both working with small groups of students and in helping to provide useful structures and tools—i.e., tables, algebraic notation, etc.

To help motivate students, encourage them to make conjectures, and also practicing breaking conjectures with counterexamples and refining conjectures to make them stronger.

Exit Tickets

While we don't ask for anything to be turned in for the rich task directly, we will give exit tickets, which students can solve in pairs and should turn in individually. These are checks to understanding that should help students track their own learning and demonstrate their progress to the teacher.

Don't hand out exit tickets until the final 10 - 15 minutes of class. Students should have plenty of time to explore the task, and only turn their attention to the exit ticket once they've already had plenty of time to explore.

I hope this curriculum helps your students remain engaged and actively learning! This is a relatively early draft, so if there's something you see that could be improved, please email me at dan@mathforlove.com to let me know.

Enjoy!

[1] Fawn Nguyen's visualpatterns.org was an inspiration for this opener, though the actual images were designed for this curriculum.

Day 1

Opener: Counterexamples
Task: 1-2 Nim
Exit Ticket: Nim explanation and 1-2-3 Nim followup

We recommend using Counterexamples as an opener today. Even though some of the examples in the writeup are designed for younger students, it's easy to challenge older students as well. Consider posing some of the following (erroneous) conjectures.

- When you multiply two numbers, the product is larger than either.
  [Counterexamples: 1×5, 0×5, $\frac{1}{2} \times 6$, $4 \times (-4)$, $\frac{1}{2} \times \frac{1}{2}$]
- The product of two numbers is always greater than their sum.
- If two shapes have equal areas, then they have equal perimeters.
- If two rectangles have equal areas, then they have equal perimeters.
- If two rectangles have equal perimeters, then they have equal areas.
  [Counterexample: 9 by 11 vs. 10 by 10 rectangles]

During the rich tasks throughout this program, students will have more success when they make conjectures and take in the information that counterexamples give when they break those conjectures.

Counterexamples

Topics: logic, deduction, mathematical argument, communication
Recommended Grades: K - 12
Materials: None
Duration: 10 - 20 minutes for basic lesson. 1-2 lessons for Pattern Block example.

Prove the teacher wrong. Rigorously.

Why We Love Counterexamples

Every kid loves to prove the teacher wrong. With Counterexamples, they get to do this in a productive way, and learn appropriate mathematical skepticism and communication skills at the same time.

It is possible to play Counterexamples with kids as young as kindergarteners as a kind of reverse “I Spy” (“I claim are no squares in this classroom. Who can find a counterexample?”). What’s great, though, is that you can transition to substantial math concepts, and address common misconceptions. Counterexamples is a perfect way to disprove claims like “doubling a number always makes it larger” (not true for negative number or 0) or sorting out why every square is a rectangle, but not every rectangle is a square. For older kids, you can even go into much deeper topics, like: “every point on the number line is a rational number.”

The language of counterexamples is crucial to distinguish true and false claims in mathematics; this game makes it natural, fun, and plants the skills to be used later. Counterexamples is also a great way to practice constructing viable arguments and critiquing the reasoning of others.

How it works

Counterexamples is a fun, quick way to highlight how to disprove conjectures by finding a counterexample. The leader (usually the teacher, though it can be a student) makes a false statement that can be proven false with a counterexample. The group tries to think of a counterexample that proves it false.

The best statements usually have the form “All _______s are ________” or “No _______s are ________.” You can also play around with statements like “If it has _______, then it can ________.” For instance:

It’s often best to start with non-mathematical examples.
- All birds can fly. (Counterexample: penguins)
- No books have pictures in them.
- All books have pictures in them.
- If something produces light, then it is a light bulb.
- If something has stripes, then it is a zebra.

Once students have the hang of it, make the examples more mathematical.

- Doubling any number makes it bigger. (Counterexample: -1 doubled is -2, which is smaller.)
- Multiplying two numbers never gives the same answer as adding them.
  (Counterexample: 2 + 2 = 2 x 2. Or 3 + 1.5 = 3 x 1.5.)
- Fractions are always between 0 and 1.
- No square has a perimeter equal to its area.
  (Note: "equal" isn't quite right, since units are different. Counterexample: a 4 by 4 square.)

Example

Teacher: I claim all animals have four legs. Who can think of a counterexample?
Student 1: A person!
Student 2: A spider.
Student 3: A fish.
Teacher: Why is a person a counterexample?
Student 4: Because it has two legs.
Teacher: Right. I said every animal has four legs, but a human being is an animal with just two legs. So I must have been wrong. What about this one: everything with four legs is an animal.
Student 5: A spider.
Teacher: A spider is an animal with eight legs, so it proves that not every animal has four legs. But I claimed that if you have something with four legs, it must be an animal. To prove me wrong, you have to give me something with four legs that isn't an animal.
Student 5: Like a table?
Teacher: Who can tell me if a table is a counterexample to my claim?
And so on.

Tips for the Classroom

1. It's good to make up false conjectures that are right for your students. But start simple.

2. For young students, use silly claims, e.g., "The only one who likes cookies is cookie monster."

3. Kids can think of their own false claims, but sometimes these aren't the right kind, and they often have to be vetted.

4. Once you introduce the language of counterexamples, look for places to use it in the rest of your math discussions.

5. You can also use Counterexamples to motivate a normal math question. Instead of saying "draw a triangle with the same area as this square," you can say, "I claim there is no triangle with the same area as this square." If students know to look for counterexamples, this will set them to work trying to disprove the claim right away.

References: http://kuow.org/post/getting-kids-interested-math-without-their-knowing

1-2 Nim

Topics: logic, patterns, addition, counting, subtraction
Materials: Counters (tiles, beans, pennies, etc.) and/or paper and pencil
Time: 1-3 lessons

You can take one or two counters from the pile. How do you get the last one?

Why We Love 1-2 Nim

Nim is fun, challenging, and rewarding for a wide range of students. Completely unlocking the game is an exciting and powerful achievement for a student. Extensions for the game abound.

The Launch

It's good to highlight a few things when you launch 1-2 Nim. First, students will win and they will lose, and it's important to remember to do both gracefully. But second, losing is better than winning, in a way, because every time you lose, you can see what strategy your opponent used to beat you, and then learn that strategy. That's how you become a N*im Master*.

Launch the game by playing a few demonstration games with students.

Instructions

Nim is a two-player game. Start with a pile of 10 counters. On your turn, remove one or two counters from the pile. You must take at least one counter on your turn, but you may not take more than two. Whoever takes the last counter wins.

Example Game

Start with 10 counters in the pile.
Player A takes 2 counters, leaving 8.
Player B takes one counter, leaving 7.
Player A takes two counters, leaving 5.
Player B takes one counter, leaving 4.
Player A takes one counter, leaving 3.
Player B takes one counter, leaving 2.
Player A takes two counters, leaving 0 and winning the game.

Play several demonstration games as needed. When the students understand the rules, have them play against each other in pairs. Students can try to challenge the teacher if they think they have a strategy that can win.

The Work

As students play, the teacher can move among them and challenge them to play, or ask them what they've noticed so far. The teacher should be able to beat the student unless the student plays perfectly. When students have played for 10 - 15 minutes or so, bring them together again and discuss what they've noticed so far. Students may have noticed that when they can give their opponent 3 counters, for example, they win. (We call this the *3-trap*, since if you can give your opponent 3 counters, you have effectively trapped them, and can win the game no matter what they do.)

Pose the central question and discuss: how can you win at 1-2 Nim?

Students may have philosophical questions related to game-playing: what does it mean to have a winning position or losing position in a game? Is there luck in the game? Where does the game go from feeling "random" to feeling like you can control it?

These are productive conversations. Before students go back to work, there are two points to underline to make the exploration productive:

1) Making the game simpler makes it easier!
   How can you make the game simpler? By shrinking the pile. Challenge the students to play you with a pile of 1 counter. Do they want to go 1st or 2nd? What about with 2 counters? It's so easy it feels like a joke, but this is what serious mathematicians do, and we get valuable information here.

2) Make a table!
   This is how the data you collect by making the game easier can actually help you. The beginning of a table might look like this.

Number of Counters	Winning Strategy
1	Go first. Take 1.
2	Go first. Take 2.
3	Go second
4	...
5	...

If students want to master the game, all they have to do is extend the table. It tells them what to do.

Prompts and Questions

The Central Question: how can you win 1-2 Nim?

Good questions for the teacher to ask students:

- What move should I (the teacher) make?
- How did you/they/I win that game?
- What do you think your/my opponent will do if you/I take two counters?
- Would you like to take back your move?
- What have you noticed about this game?

Possible student conjectures (all interesting, all false or incomplete) that may arise:

- Whoever goes first wins.
- Whoever goes second wins.
- Odd vs. even numbers of beginning counters determines your strategy.
- It matters/doesn't matter what you do until there are less than six counters in the pile.
- Whoever can give their opponent four open counters wins.

The Wrap

It may be wise to end class without a conclusion, depending on where students are, and discuss again on a subsequent day. Send them home to try out the game against friends and family, and refine their strategy.

When students are ready, discuss strategy—do students have any ideas of how to win, regardless of the size of the pile? Once they share, ask if anyone would like to try another game against you. Let them get advice from their peers (students can quietly raise one or two fingers to suggest what they think should happen). Can they beat you?

The major breakthrough is that the table actually tells you what to do. Have a student share a table and discuss its contents. What patterns do students see? Students should be able to defend these results.

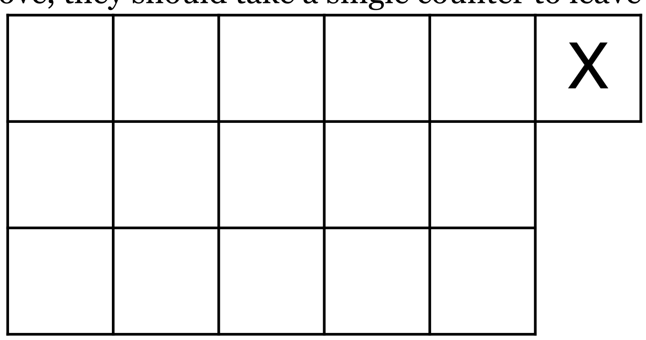
Number of Counters	Winning Strategy
1	Go first. Take 1.
2	Go first. Take 2.
3	Go second.
4	Go first. Take 1.
5	Go first. Take 2.
6	Go second.
7	Go first. Take 1.

You can play students with the table visible. On your turn, look at the table and talk out loud what you should do, i.e., “There are 7 counters, and its my turn. So the table says, take 1.” The takeaway here is that the table is literally instructions for winning.

Students will also, hopefully, notice the pattern in the table. (Go first, take 1. Go first, take 2. Go second.) It looks like the “3-trap” actually extends to a “6-trap” and a “9-trap,” and so on. In other words, the winning strategy might be as simple as: on your turn, give your opponent a multiple of three. Does that really work? Challenge the class to a game with 25 counters. Let students discuss their strategy, and then choose a student to play you. Can they win?

Finally, when students can articulate a winning strategy and successfully beat you, there’s still a question of *why* this “give your opponent a multiple of three” strategy succeeds. Here’s a sketch of an argument that shows why it does.

Arrange, let’s say, 16 counters in a 3 by 5 array, with one extra counter. When the first player makes their move, they should take a single counter to leave a 3 by 5 array.



Whatever their opponent does, they’ll always be able to give back a 3 by *something* array. Try it out! So the choice of arrangement of tiles actually makes the argument clear.

There’s a moral to this exploration: the way to *become* a master of nim, or any game, is to apply mathematical rigor and organization. In other words, thinking and working mathematically makes you powerful.

To end, you can challenge students with a couple of potential extensions.

Variations

1) What’s the right first move if you’re playing 1-2 Nim with 150 counters? What about 542 counters?
2) Try 1-2-3 Nim: players may take one, two, or three counters per turn. How do you win this game?
3) Try 1-2-3 Poison: Whoever takes the last counter <u>loses</u>.
4) What about 1-3-4 Nim? Players may take one, three, or four counters, but NOT 2.

Tips for the Classroom

1. Demonstrate the game with student volunteers for at least three games (or many more!), until you are certain everyone understands it and is excited to play.

2. When demonstrating 1-2 Nim, narrate the game out loud, using mathematical language, and leaving empty space for students to chime in: "My opponent just took 2 leaving... [wait for students] 5 in the pile. Who has a advice for what I should do next?"

3. Remind students that they will lose many games as they play, and that every loss is an opportunity to learn. Can they steal the strategy of the person who just beat them? Point out how students are trying out new strategies as they play you in demonstration games.

4. As students play each other, circulate to see what strategies they are developing. Challenge them to play you, and see if they can beat you.

5. Encourage student conjectures, but do not call them as true or false. Challenge students to break their own conjectures.

6. A big moment in taking ownership of the game is to change the size of the pile. Making the pile smaller makes it easier to understanding and win. Making it bigger makes it more challenging.

7. We use the term "3-trap" to describe how you trap your opponent by giving them a pile of three counters. Understanding how to win boils down to understanding what pile sizes you want to leave your opponent with.

8. There are two incredibly powerful approaches to solving Nim. The first is to simplify. How could the game be easier? What if the pile had only one counter? From this place of almost absurd simplicity, we slowly raise the difficulty. What about two counters? Three counters?

9. The second approach is to organize the data in a coherent way. A table does this very nicely.

10. If student want to play three-player, keep in mind that we discourage it. Normally trying out different numbers of players is a great impulse. In Nim, it leads to spoilers, who can't win, but can choose who does win, which diffuses the mathematical tensions in the game.

11. Optional homework: have students teach 1-2 Nim to a friend or family member.

Name______________________________________

Exit Ticket

Discuss with a partner, and turn in your own work.

Part 1.
Write down your explanation for how to win 1-2 Nim.

Part 2.
In the game of 1-2-3 Nim, you can take 1, 2, or 3 objects from the pile each turn. Is 15 a winning or losing position in 1-2-3 Nim? If it is a winning position, what is your winning move? Explain how you know.

Day 2

Opener:	Visual Patterns
Task:	Star Polygons
Exit Ticket:	18 points in a circle

For these visual patterns, there are three questions.
You can answer them with equations.

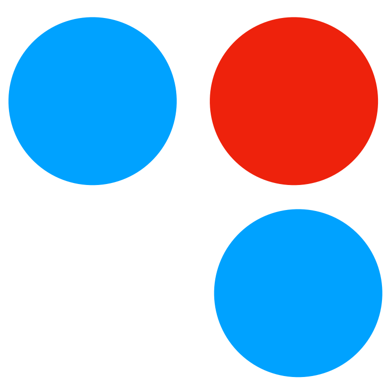
1) How many red in the nth state?

2) How many blue in the nth state?

3) How many total in the nth state?

Hint. Sometimes answering these questions in different orders first will be easiest.

For more like these (in one color), see visualpatterns.org.



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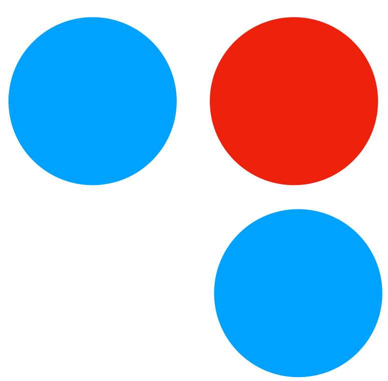

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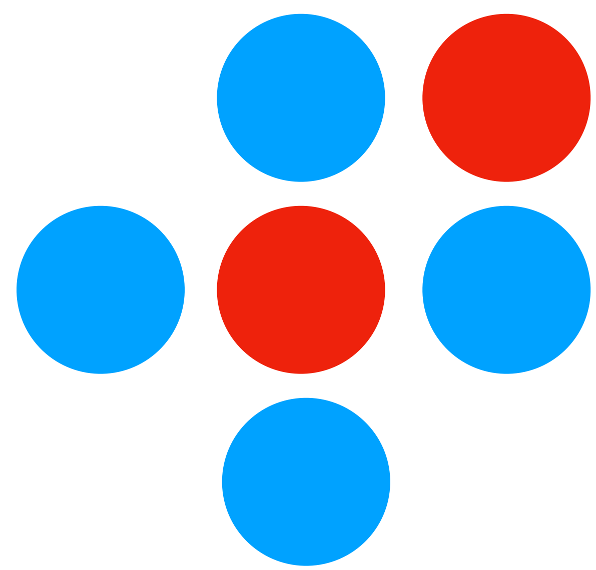

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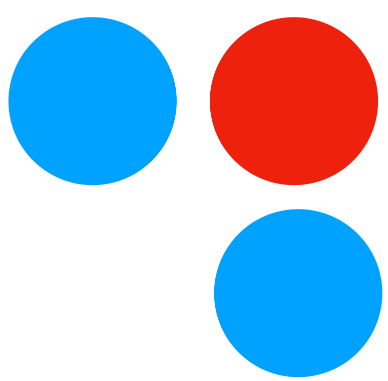

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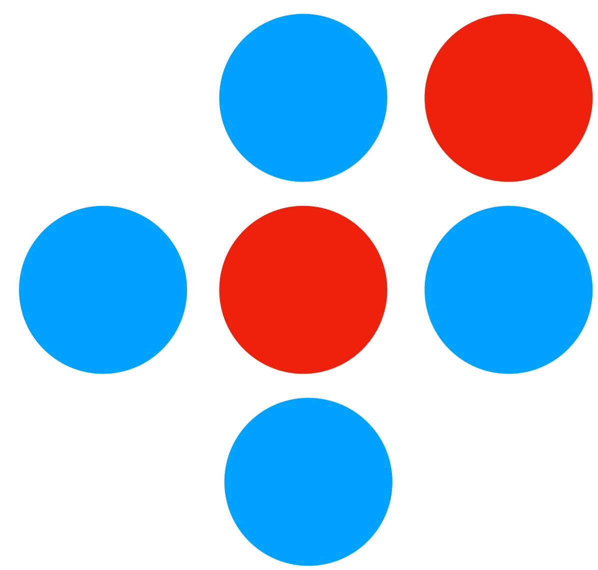

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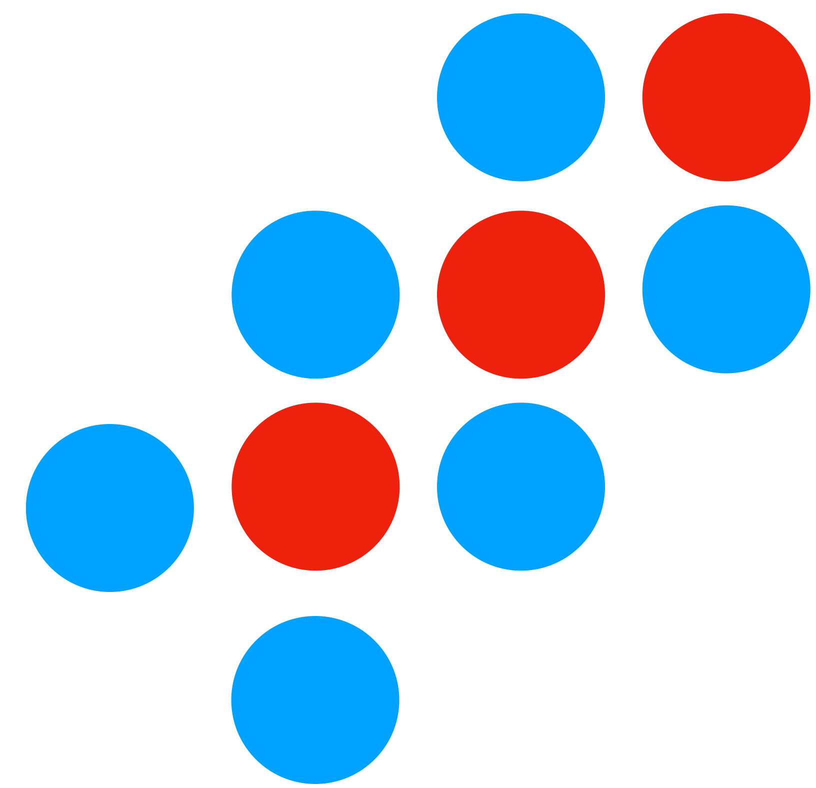

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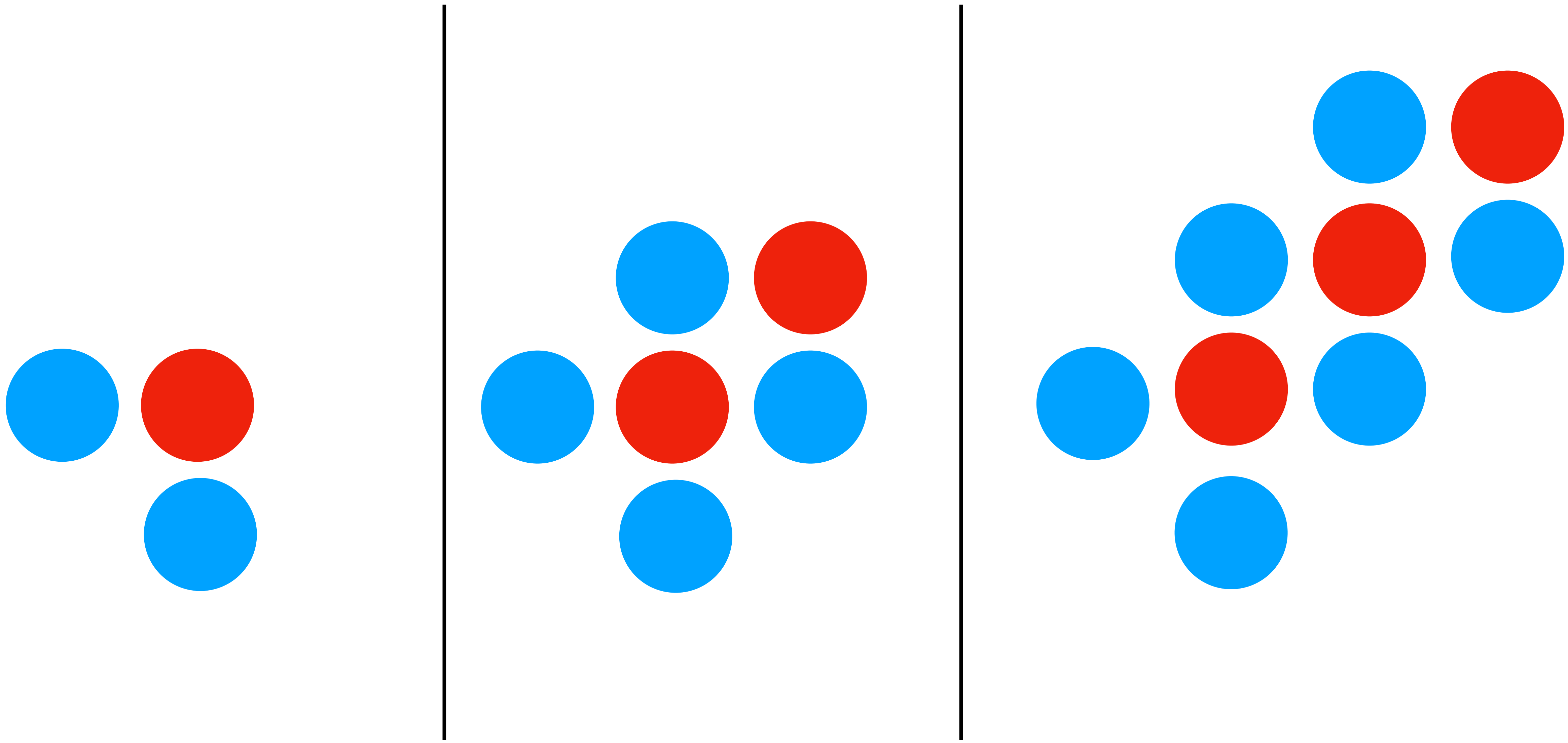


Stage	Red	Blue	Total
1	1	2	3
2	2	4	6
3	3	6	9
n	n	2n	3n

Name________________________

Star Polygons

Concepts: Patterns, factors & multiples, primes, ratios & proportion,
Materials: Worksheets with equally spaced dots, scratch paper, Geogebra or other computer geometry program
Common Core: 4.OA.5, 4.G.1, 5.OA.3, 5.G.4, 6.RP.1, 6.RP.1, 7.RP.2, 7.RP.3, 7.G.2, MP1, MP2, MP3, MP5, MP6, MP7, MP8

A rich environment to explore, and a perfect introduction to the art of making and breaking counterexamples.

Why we love star polygons

These geometric designs are beautiful and fascinating to study. Students can get involved almost immediately, and there's plenty to discover as they dig in.

The Launch

Show the accompanying video.

If you launch without the video, we recommend Counterexamples as a warm up.

Start with the eight-pointed star, and ask for a number from the class. (Three is a good number to demonstrate with.) Starting at the top, connect a dot to the dot "three away," and then repeat until every dot is connected. That's an interesting discovery!

Conjecture. No matter what number we picked, we would have hit every dot.

Students will probably offer 0, 2, or 4 as a counterexample. Indeed, all these work. After a little more work of seeing which numbers lead to all the dots being hit, we might arrive at the following:

Conjecture. Odd numbers will hit every dot. Even numbers won't.

This is true (for 8 dots). A piece of vocabulary here: if the "connection rule" leads to us connecting up all the dots, we call the finished design a "star polygon." You can take a minute to ask students if they see anything wrong with this name. In particular, star polygons aren't *polygons*, technically, since they involve crossed lines. Still, that's what they're called!

Students should, hopefully, have the hang of how the connection rules work, though it's easy to make mistakes. (Demonstrating mistakes is a good idea, in fact.)

You can pose: will this conjecture hold for different numbers of dots? Let's try to break it! If we see other patterns, we can start to pose new conjectures too. And indeed, that will be the main work for today. Pass out the sheets and let student explore.

Name________________________

Prompts and Questions

- What conjectures have you written down so far?
- What conjectures have you disproved?
- How could you organize your data so it is easier to read?
- Would your idea work on 11 dots? Try it!

The Work

As the students work, you'll likely need to handle the following issues that may come up:

1. A few students may not understand the process, or the "connection rule." Find them and help them get sorted out.
2. Once everyone is productively searching, you'll want to call the class together occasional to
   i. Let students share conjectures (and counterexamples), and
   ii. Share a good way to organize data to look for patterns.

Probably the best structure is the simple chart:

Number of dots	Connection rules that lead to a star polygon	Connection rules that don't lead to star polygons
6	1, 5	2, 3, 4, 6
7	1, 2, 3, 4, 5, 6	7 (sidenote: should we call this a connection rule?)
8	1, 3, 5, 7	2, 4, 6, 8
9	1,2,4,5,7,8	3,6,9
10		
11		

Once students use this kind of organization, their ability to make coherent conjectures and find counterexamples will make a leap forward.

The Wrap (and extensions)

This lesson can go 1-3 days. If you just spend a single day, don't expect students to come to a perfect conclusion. However, you can bring them together and let them discuss what conjectures are still standing, and what their best current guesses are for what's really going on. Some conjectures that may come up on the first day:

Conjecture. If the number of dots is odd, every even connection rule will produce a star polygon.
Note: This is false. Letting the number of dots D = 15 and the connection rule C = 6 produces a counterexample.

Name________________________

Conjecture. If D is even, and C is odd, then you get a star polygon.
Note: Also false. D = 10 and C = 5 gives a counterexample.

Conjecture. If the number of dots D is prime, any connection rule from 1 to D-1 will hit every dot, producing a star polygon.
Note: This is true.

Conjecture. There's a symmetry in connection rules: connecting k and D-k dots both produce the same picture (i.e., with 5 dots, connection rules 2 and 3 both produce a 5-pointed star).
Note: Also true.

Conjecture. If C/D is a fraction in lowest terms, then they produce a star polygon.
Note: This is true, but subtle to prove. You'll need to get into the extensions for students to see why this is true, which will probably take addition days. But they'll be rich, mathematically!

Several extensions are possible, depending on students' insights.

Extension 1: When do you get a square? A triangle? A five-pointed star?
Looking at particular shapes that emerge from dots and skip rules is a great way to connect to issues of ratio and proportionality. When, for example, do we end up drawing a five-pointed star? Certainly with five dots (D = 5) and connection rule 2 or 3 (C = 2,3). But also D = 10 and C = 4, 6. And D = 15 and C = 6, 9. Indeed, when the proportion D:C = 5:2 or 5:3, we get the 5-pointed star. Exploring this connection will likely allow students to delve into concrete and abstract understandings of ratio and proportion, especially as they test their more and more general conjectures of this nature. They can continue to make conjectures until, ideally, they can articulate why this ratio perspective makes sense.

Extension 2: A complete accounting for when star polygons occur and when they don't. When do we hit every point? If we apply our understanding from extension 1, we can see that when D and C have any common factor (greater than 1), there must be dots we miss. But what about when they don't have a factor in common? In this case, it turns out we do hit every point. Proving that this is indeed the case is an excellent project for older or more advanced students. For even more advanced students, counting the number of C that create star polygons for a given D is a very interesting and challenging project. (The answer is known as the *totient* or *Euler phi function*.)

Extension 3. What if C is larger than D? Trying connection rules greater than the number of dots leads to a potential exploration of modular arithmetic.

Extension 4. What are the angles of a star polygon? Exploring the angles leads to an entirely different, beautiful theory.

Name________________________

Tips for the Classroom

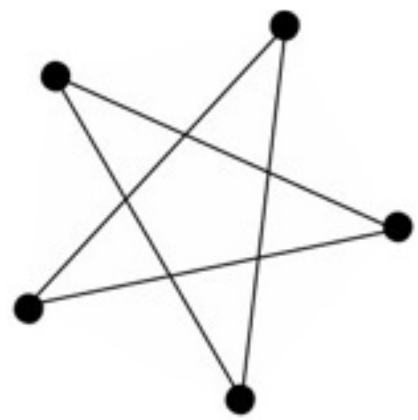
1. Use Geogebra or another computer program to draw equally spaced dots and the star polygons that result.
2. Helping students organize their search and record their data in a table will be one of the most fundamental ways to help them come up with conjectures.
3. Searching for counterexamples is a great job for students who might be temporarily aimless.
4. Even if your students are capable of using algebra or abstraction, make sure they defend their thinking with concrete examples.

Name______________________

Star Polygons

The five-pointed star below was made by starting with 5 evenly spaced points and connecting every 2nd point. We say we've used the "connection rule of 2."

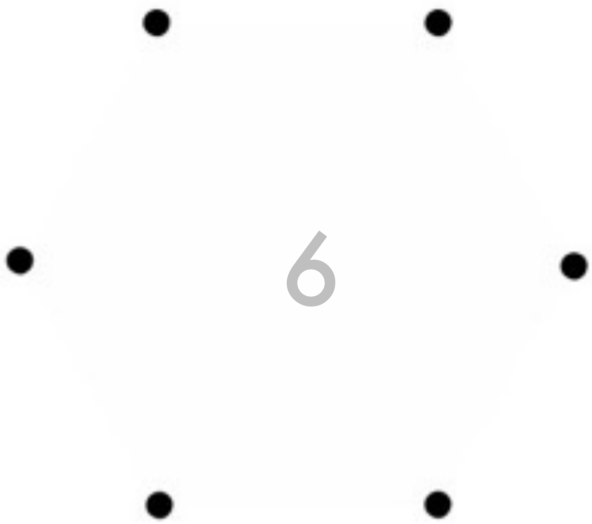
It's called a *star polygon* when you hits **all the points** you started with in one continuous loop.

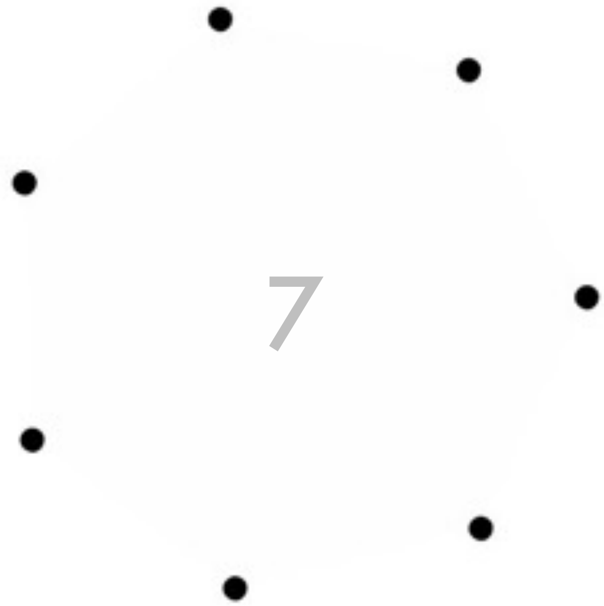


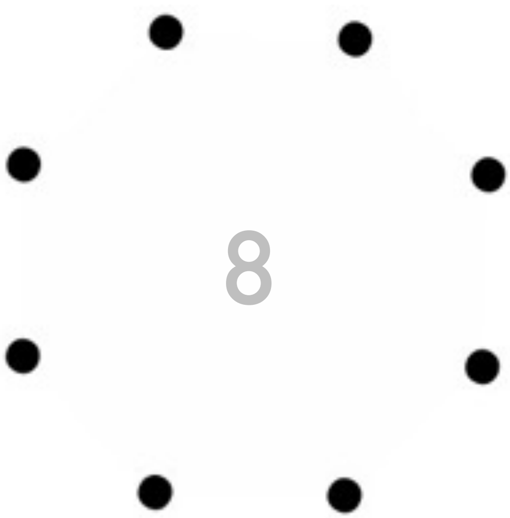
Working with a partner, experiment with building star polygons or other shapes by connecting regularly spaced points.

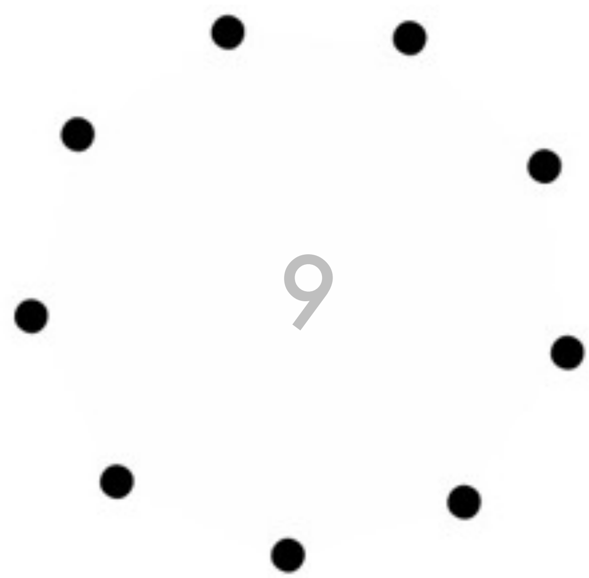
Write conjectures as you go. Which conjectures can you break? Which seem to hold?

Name______________________









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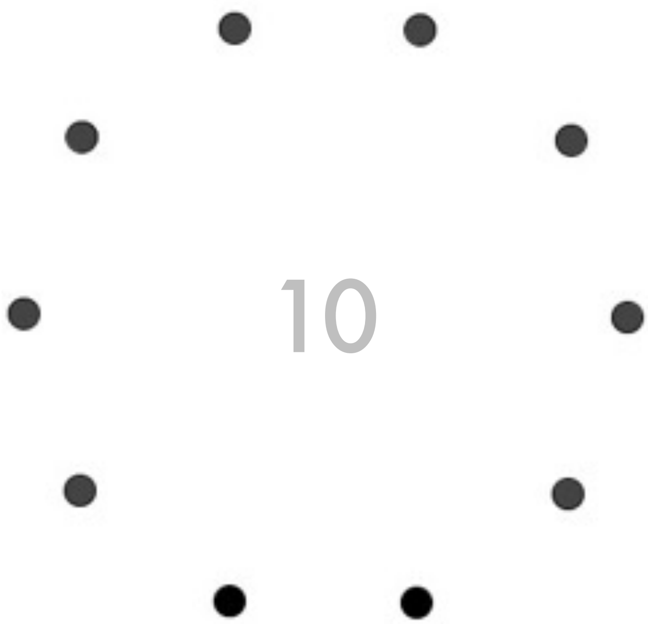

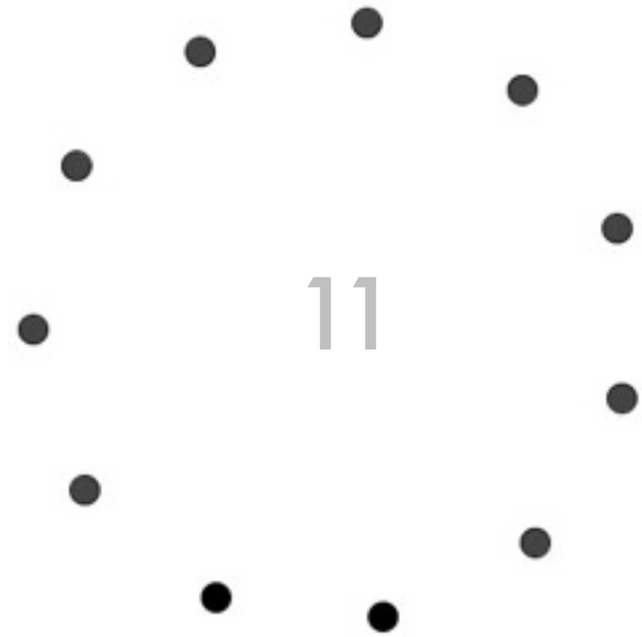

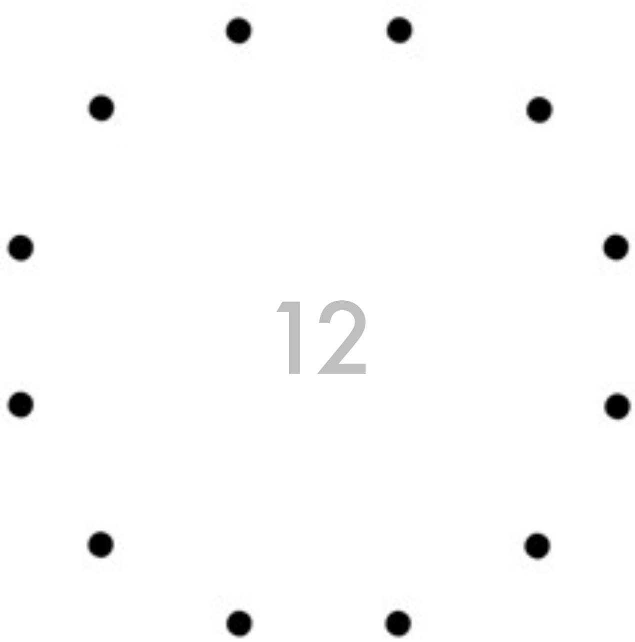

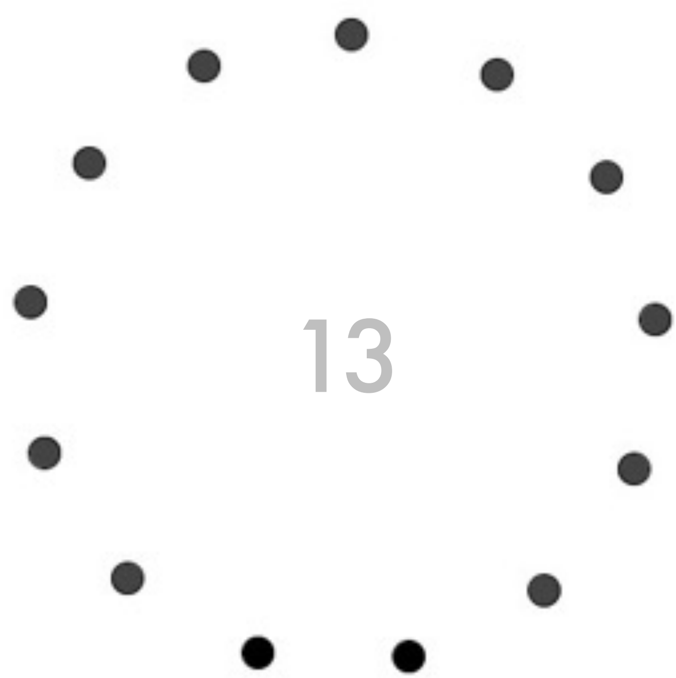

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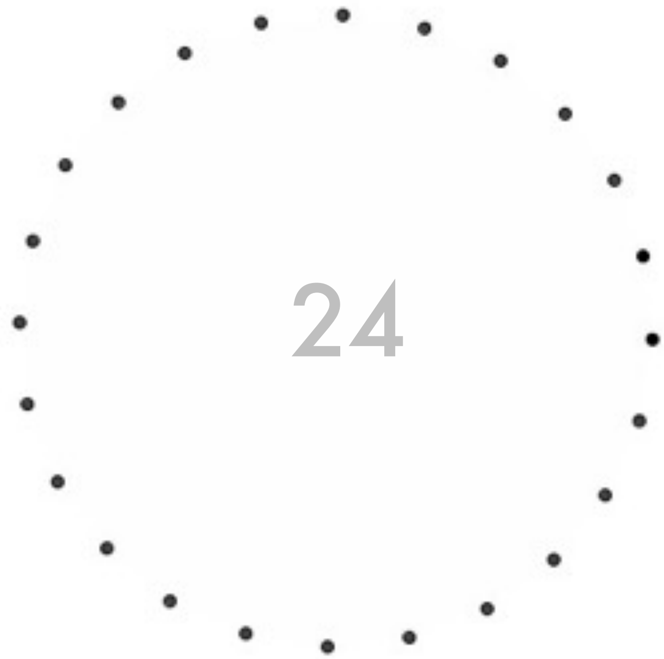

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Exit Ticket

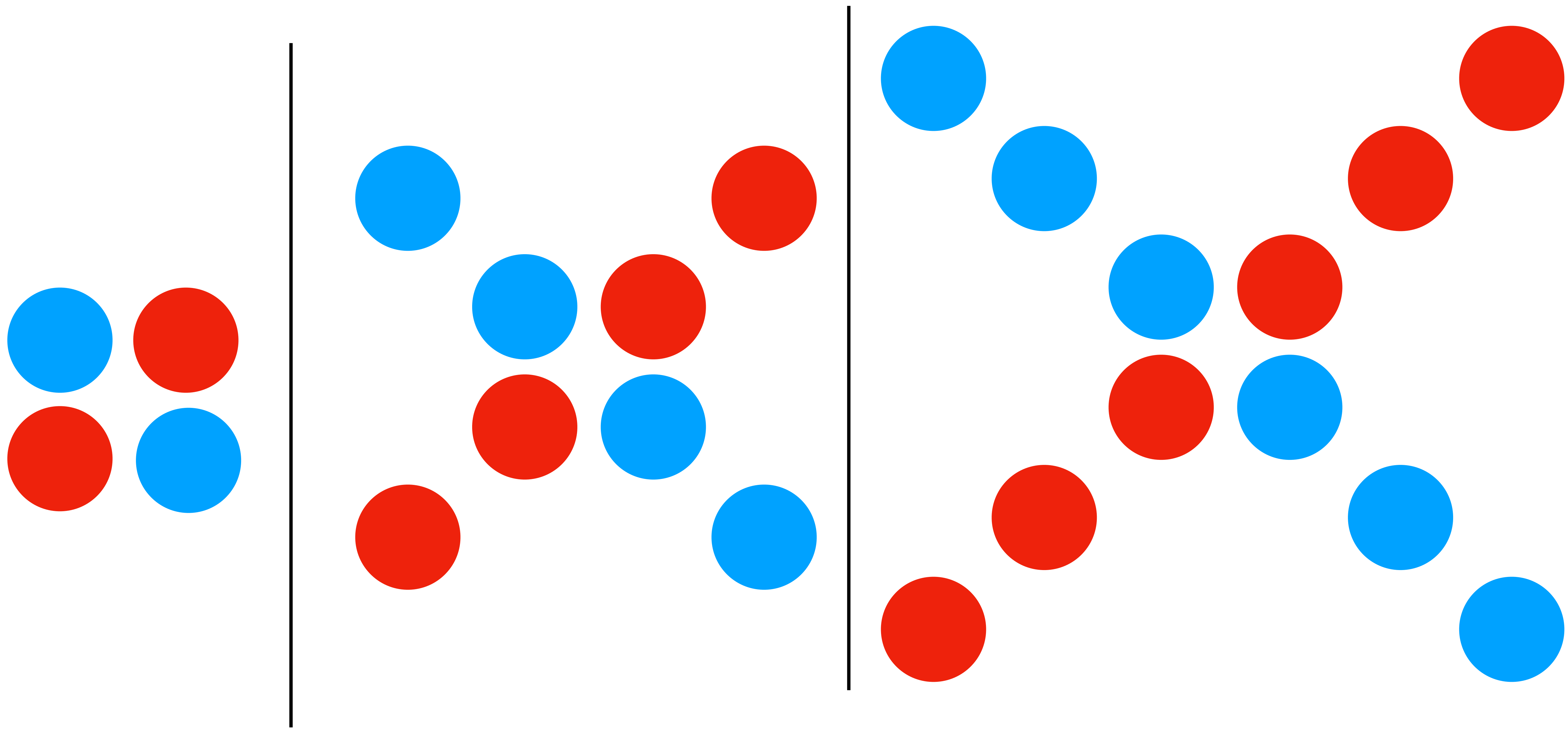
Discuss with a partner, and turn in your own work.

Part 1.
Write down one conjecture you think is probably true about star polygons.

Part 2.
Consider 18 dots in a circle. For the jumps between 1 and 17, what jumps will hit every dot and produce star polygons? What do these jump numbers have in common? Which jumps will miss dots?

Day 3

Opener:	Visual Patterns
Task:	Prime Climb Chart Coloring
Exit Ticket:	The coloring for the number 105



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Prime Climb Chart Coloring

Topics: Multiplication, division, addition, subtraction, multi-step problems, factoring
Materials: Prime Climb coloring charts, crayons or colored pencils
Time: 1 lesson

What patterns can you find in the hundred chart?

Why We Love Prime Climb Chart Coloring

The image that launches this lesson and discussion is one of the richest we know. The colored chart contains color-coded information about factors and multiples, and illuminates connections between them.

Launch

Show students the chart with the coloring to 20, and ask them what they notice, and what they wonder. Give students some time in pairs to discuss or write down their ideas. Then discuss as a class. In particular, what are student conjectures to explain the way the numbers are colored?

Possible student observations might include:
- Every even number has an orange segment.
- If there is green in the numbers you can reach it if you skip count by 3.
- The circles with blue are numbers end in 5 or 0.
- The circles with red are prime numbers.

Questions these conjectures leave unsolved:
- Why does 4 have two orange parts?
- Why does 8 have three orange parts?
- If prime numbers are red, why isn't 7 red?

The Work

Once students have discussed some ideas for how the coloring works, challenge them to color in the numbers 21 - 30 so that it extends the pattern. Let students work in pairs. If they are stuck after five minutes or so, you can gather the class together and discuss how to color 21. If the patterns will still work, 21 should be colored green, because it can be reached if you skip count by 3s. It should also be purple, since you can reach it if you skip count by 7s. Another rationale for the coloring: 3 is green, 7 is purple, and 3 x 7 = 21, so 21 should be green and purple.

Once students have the hang of how the coloring works, let them work on their own again. They can color in as much of the chart as they can, but getting to 30 is a good initial goal. In general, multiplying and dividing or skip counting is the key to understanding how the coloring works.

Prompts and Questions

- Look at just one color at a time. What's happening with orange? What's happening with blue?
- What numbers have blue? (5, 10, 15, 20.) What do you think comes next in that pattern? So probably 25 and 30 will have a blue part colored in, right?
- Which numbers have two or more orange parts colored in (4, 8, 12, 16, 20)? What pattern do you see in those numbers?
- If I wrote out some multiplication problems like 2 x 5 = 10, 3 x 5 = 15, 4 x 5 = 25, what do you notice about the colors of all the numbers in the equations?

The Wrap

Pick a few numbers that everyone has at least thought about: 28, 29, 30, for example, or 22, 23, 24 if students haven't gotten that far. Let students defend their choices for coloring. Why should 23 be red? Why does 24 have three orange segments and one green segment? The teacher can use the chart colored up to 60 as a reference, but it's best if students can argue why a given coloration works, and convince other students based on multiplication/skip counting arguments.

Don't expect students to finish the entire chart in one lesson. They can come back to it in the future.

Tips for the Classroom

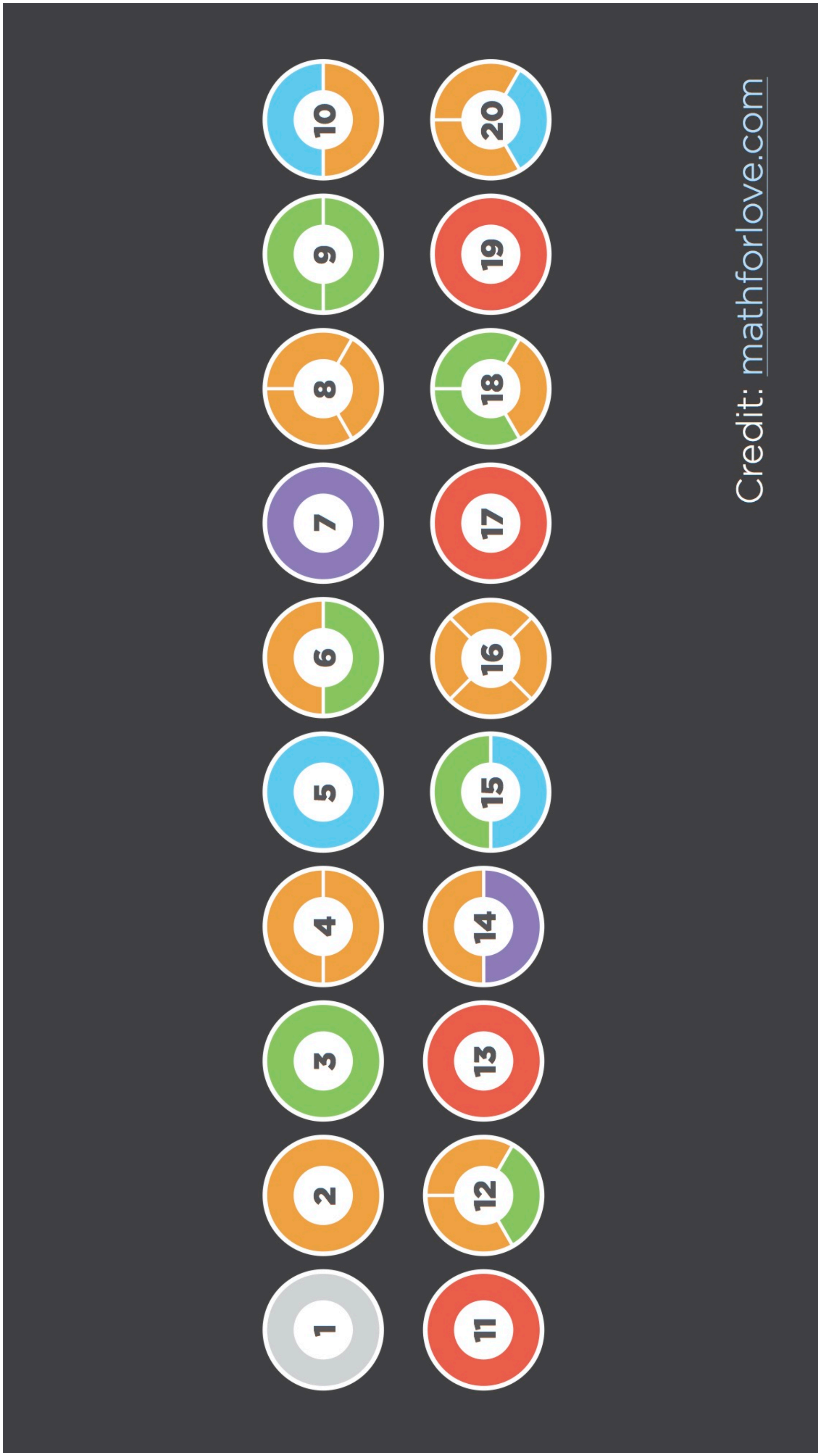
1. If students have a hard time articulating their conjectures, encourage them to zero in on a single color at a time. What's the pattern with the blue numbers? What about orange, or green?

2. Students may want to work number by number. They also may want to use the patterns in the colors to, say, color a blue segment in all the multiples of 5. Both are good strategies.

3. It's a good idea to establish some conjectures about how to check if a number is a certain color: for example, if you can skip count to some by a smaller number, then all the colors of the smaller number are in the bigger number too. This gives students a way to check their work.

References

1. mathforlove.com/lesson/prime-climb-color-chart
2. primeclimbgame.com/teach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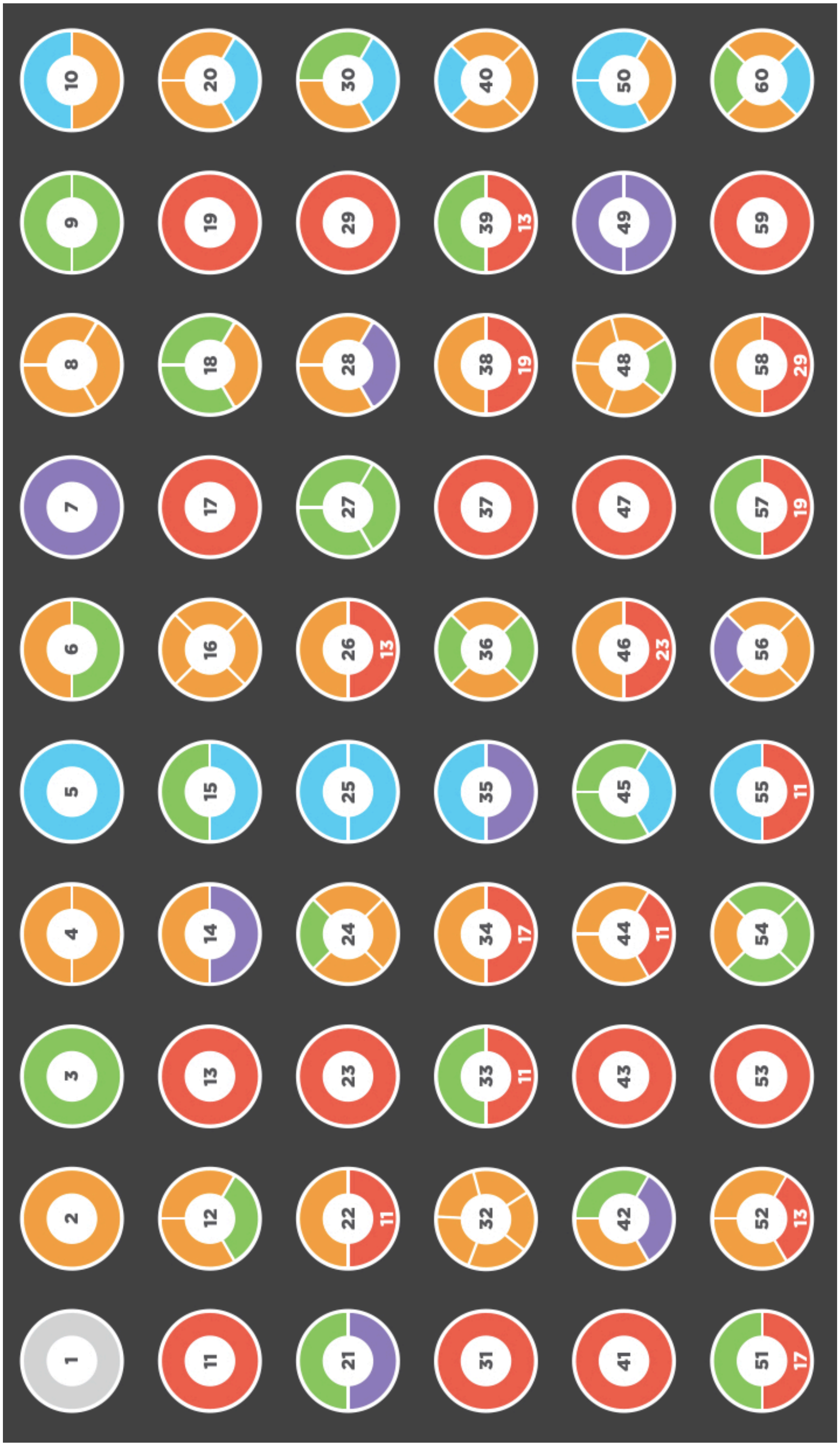
Credit: mathforlove.com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61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

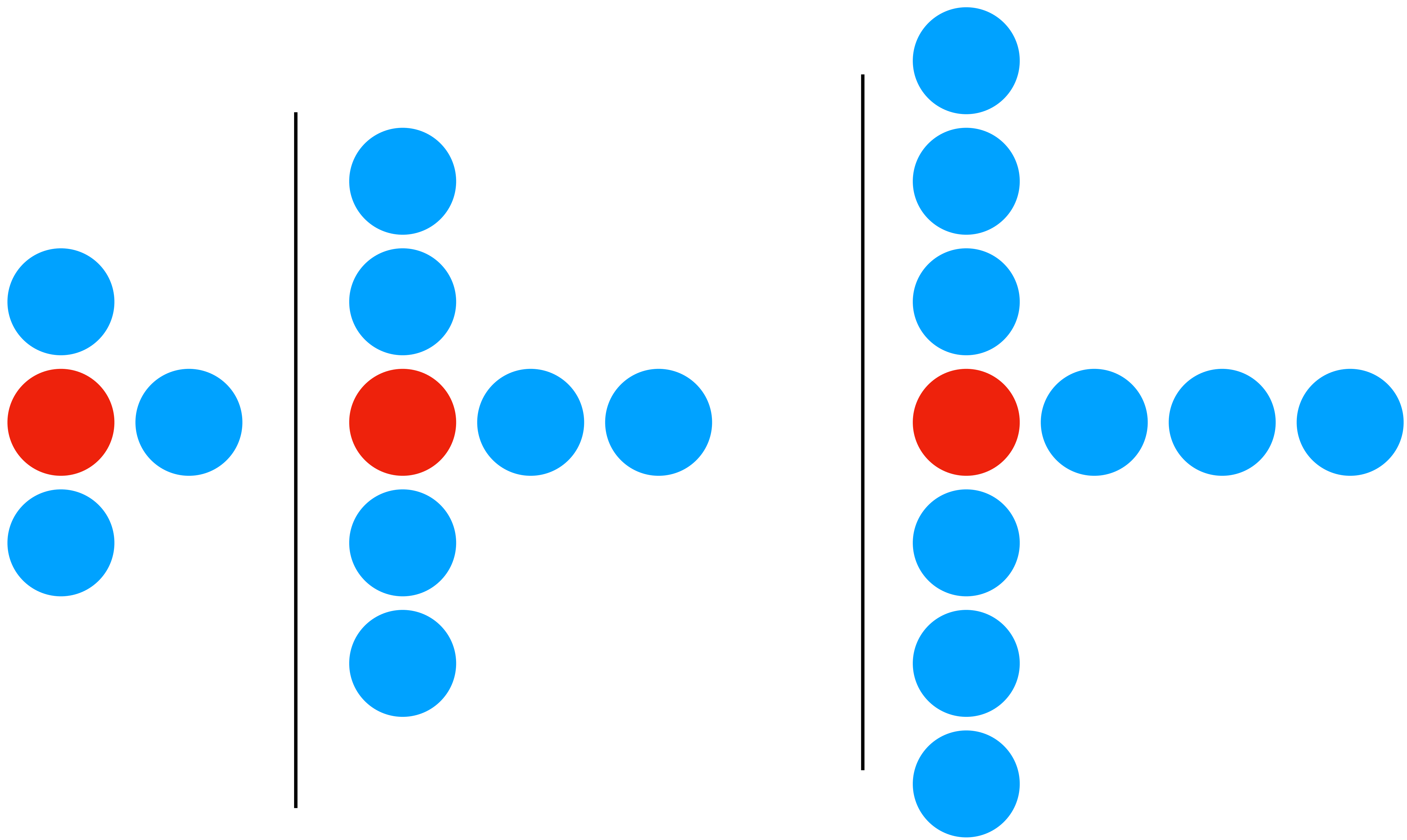
PRIME CLIMB

Exit Ticket

What is the correct coloring for 105?
Defend your answer.

Day 4

Opener:	Visual Patterns
Task:	Beat the Tax Collector
Exit Ticket:	What’s the worst first move in Beat the Tax Collector, and why?



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Beat the Tax Collector

Tax Collector is played like this: start with a collection of paychecks, from \$1 to \$12. You can choose any paycheck to keep. Once you choose, the tax collector gets all paychecks remaining that are factors of the number you chose. *The tax collector must receive payment after every* move. If you have no moves that give the tax collector a paycheck, then the game is over and the tax collector gets all the remaining paychecks.

The goal is to beat the tax collector.

Example.
Turn 1: Take \$8. The tax collector gets \$1, \$2, and \$4.
Turn 2: Take \$12. The tax collector gets \$3 and \$6 (the other factors have already been taken).
Turn 3: Take \$10. The tax collector gets \$5.

You have no more legal moves, so the game is over, and the tax collector gets \$7, \$9, and \$11, the remaining paychecks.

Total Scores.
You: \$8 + \$12 + \$10 = \$30.
Tax Collector: \$1 + \$2 + \$3 + \$4 + \$5 + \$6 + \$ 7 + \$9 + \$11 = \$48.

Questions.

1. Is it possible to beat the tax collector? If so, how? What is the maximum score you can get?

2. If you play with a different number of paychecks, when is a win possible?
   With \$1? \$1 and \$2? \$1 to \$3? \$1 to \$4? ... \$1 - \$24? \$1 - \$25?

3. Solve the last problem for numbers greater than \$25. Is there a general rule that determines for which numbers it will be possible to beat the tax collector?

4. What is the largest number you can find where it is possible to beat the tax collector?

\$1	\$2	\$3	\$4
\$5	\$6	\$7	\$8
\$9	\$10	\$11	\$12
\$13	\$14	\$15	\$16
\$17	\$18	\$19	\$20
\$21	\$22	\$23	\$24

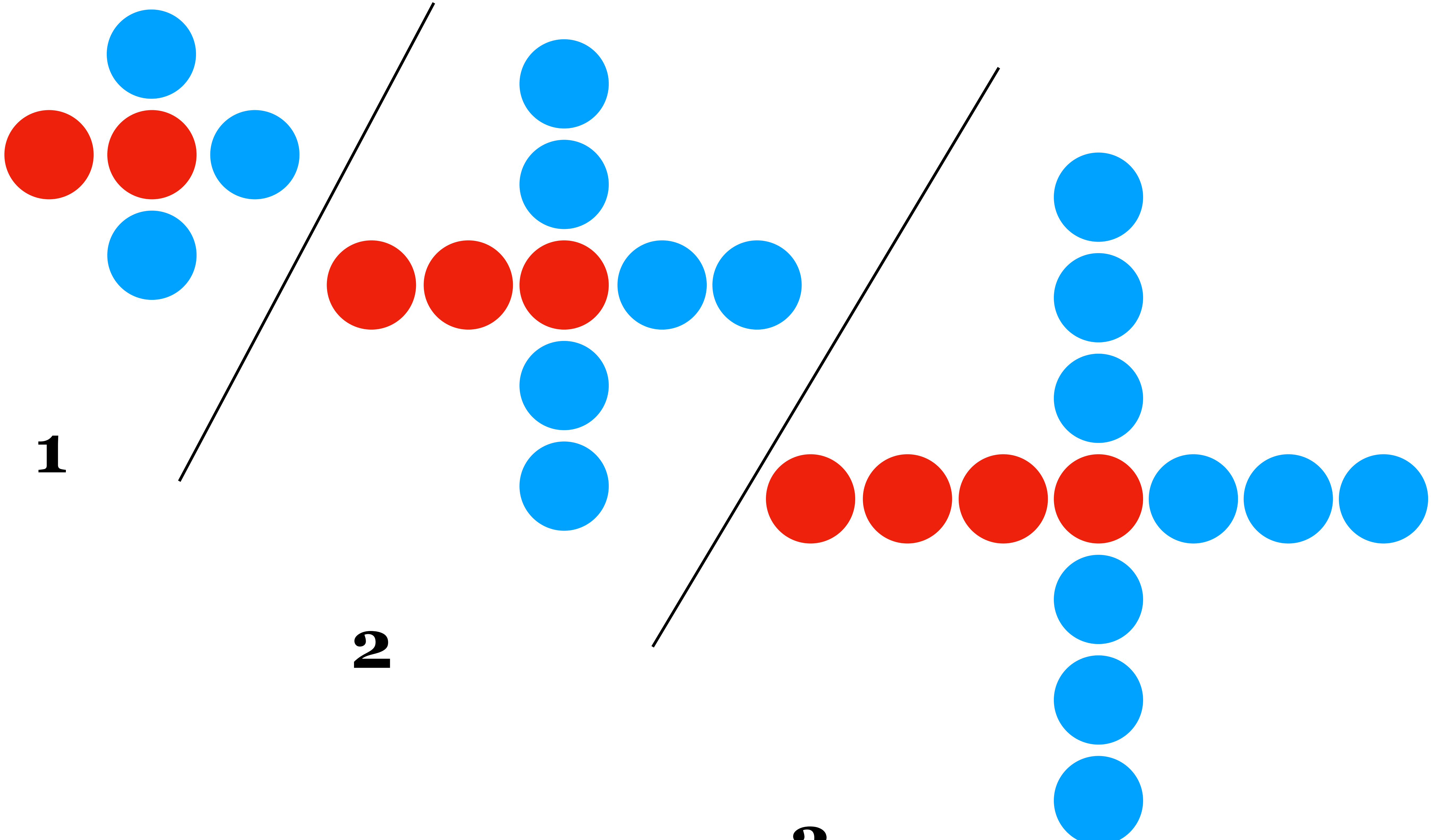
Exit Ticket

What is the **worst** first move you can make in a game of Beat the Tax Collector?

Why is it the worst?

Day 5

Opener:	Visual Patterns
Task:	Math Magic Trick
Exit Ticket:	An algebraic description for the math magic trick, with 2s and with 3s.



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A Math Magic Trick

Topics: Addition, Subtraction, Multiplication, Division, Problem Solving
Materials: Scratch paper and pencil, or white boards, Cuisenaire rods (optional)
video launches (optional):
Part 1: **https://www.youtube.com/watch?v=n2RTjqHUXGQ**
Part 2: **https://www.youtube.com/watch?v=HMSmt5yOl6E**
Duration: 1 - 2 lessons

How does it come out the same every time?

What we love about a Math Magic Trick

This deceptively simple trick can differentiate broadly. Some students can use it to practice arithmetic or get practice with arithmetic for whole numbers, or for harder topics like say subtracting negative numbers or dividing fractions.

The Launch

The teacher explains the trick to the class. Every student follows the instructions:

- Pick a whole number between 1 and 10.
- Add 2.
- Multiply by 2.
- Subtract 2.
- Divide by 2.
- Subtract your original number.

Everyone's final answer will be 1 (assuming they didn't make any arithmetic mistakes).

Big Question 1. Will this work for number that aren't whole numbers between 1 and 10? What numbers will it work for?

The Work

For most students, trying to break this pattern is a great challenge. They can try big numbers, negative numbers, decimals, fractions, or other numbers on their own or with help from the teacher. Lo and behold, everything they try will work out.

The teacher can collect conjectures about what numbers won't come out to 1 when you apply the rules of the trick. Keep checking in to challenge students with other students' ideas, or help out with arithmetic that might be a bit beyond the student comfort level.

Prompts and Questions

- What's a number you think would break the pattern?
- Do you think 17 will work? Let's try it together.

- What’s a number that might not work? Let’s try to write some guesses down, so you have a few to try.
- Have you tried really large numbers? What about 40,156?
- Do you think negative numbers will work? What about -5?
- Do you think fractions will work? What about $\frac{1}{2}$? What about $3\frac{2}{3}$?
- Do decimals work? What about 6.7?
- What’s the weirdest number you can imagine trying? What about π?
- Why do you think it will always work? Is it just because it has so far, or do you actually have a reason based on what we’re doing?
- I wonder if we could use a bar model/algebra to help.
- (for students who have solved the problem completely) What if you change all the 2s to 3s? Will it still work?

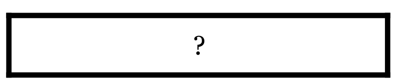
The Wrap

After they’d done a multitude of examples, we have the next question:

Big Question 2. Why does this always work?

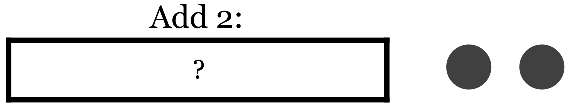
Taken at face value, this seems like an impossibly hard question. To show it always works, you need to test the trick for *every* number, and that means an infinite amount of work. How can you prove things like this are *always* true?

Depending on time, you may choose to let this question hang and pick it up when you return to this topic in a future lesson. You can also use a bar model or Cuisenaire rods to solve it. Choose a Cuisenaire rod or draw a bar model and assume this is a number, except we don’t know what number it is yet.

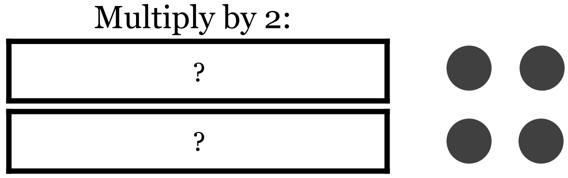


Apply the trick to the missing number, using counters to stand in for units.

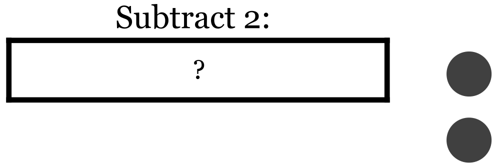
Add 2:



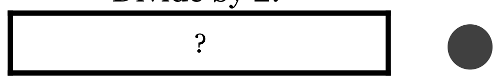
Multiply by 2:



Subtract 2:



Divide by 2:



Subtract your original number:

And here's the power of algebraic thinking made visible! No matter what that original Cuisenaire rod was worth, you always end up with 1 at the end. By leaving it at a question mark, we've actually managed to check every number at once!

This same work can be accomplished using algebra. For students who are not yet comfortable with algebra, this magic trick offers another opportunity to grasp the idea. For example, a common breakthrough is to try the trick with pi as the starting number. Since, as many students know, the digits of pi go on forever without repeating, we can't really write it out. But maybe we can just use the letter! This gives us:

π (pick a number)
$\pi + 2$ (add two)
$2\pi + 4$ (multiply by 2… this step can be a little tricky to those who don't know algebra)
$2\pi + 2$ (subtract 2)
$\pi + 1$ (divide by 2)
1 (subtract your original number)

Note: Video on this transition available at **https://youtu.be/HMSmt5yOl6E**

But here's the big idea of algebra: we didn't need to know anything about that π symbol to find out that the answer was 1. In fact, π could have stood for *any* number, and the answer would have been the same. Wait… if it could have stood for any number that the trick would still have given the answer 1, then the trick works for any number! This is actually an algebraic proof, and it gives an argument that the trick works. Always.

It may be nice to mention that folks will commonly use x rather than π.

x (pick a number)
$x + 2$ (add two)
$2x + 4$ (multiply by 2… this step can be a little tricky to those who don't know algebra)
$2x + 2$ (subtract 2)
$x + 1$ (divide by 2)
1 (subtract your original number)

From here, generalizing the trick takes us into natural extensions, which can be stand alone lessons for the future, or simply open questions for students to mull over on their own.

Big Question 3. What happens if you tinker with the trick?

A natural way to tinker is to double all the numbers.

<u>Variation</u>
Pick a number.
Add 4.
Multiply by 4.
Subtract 4.
Divide by 4.
Subtract your original number.

Do you always get the same answer?

Of course, students may have other ideas of how to tinker. Figuring out the relationship between what's changed in the trick and how the answer is affected is a great exploration, and students can go to town on it.

Tips for the Classroom

1. The main trick to this one is not to get students out of trouble too early. As long as they're confounded by the fact that the answer is always the same, you can keep goading them into trying another example (will it work for 5 halves? 137? -4.17?) and underlining how difficult the task of testing every number is. Don't rush students who aren't ready for the conceptual leap, and keep priming them to be ready for it when it comes.

2. Students will inevitably make arithmetic mistakes and think they've found a counterexample that doesn't come out to one. Have students check the work of others when they think they've found numbers that break the trick.

Maths Magic Trick

Pick a whole number between 1 and 10.

Add 2.

Multiply by 2.

Subtract 2.

Divide by 2.

Subtract your original number.

Your answer is…

Maths Magic Trick Challenge

Pick *any* number.

Add 2.

Multiply by 2.

Subtract 2.

Divide by 2.

Subtract your original number.

Find a number where the answer WON'T be 1.

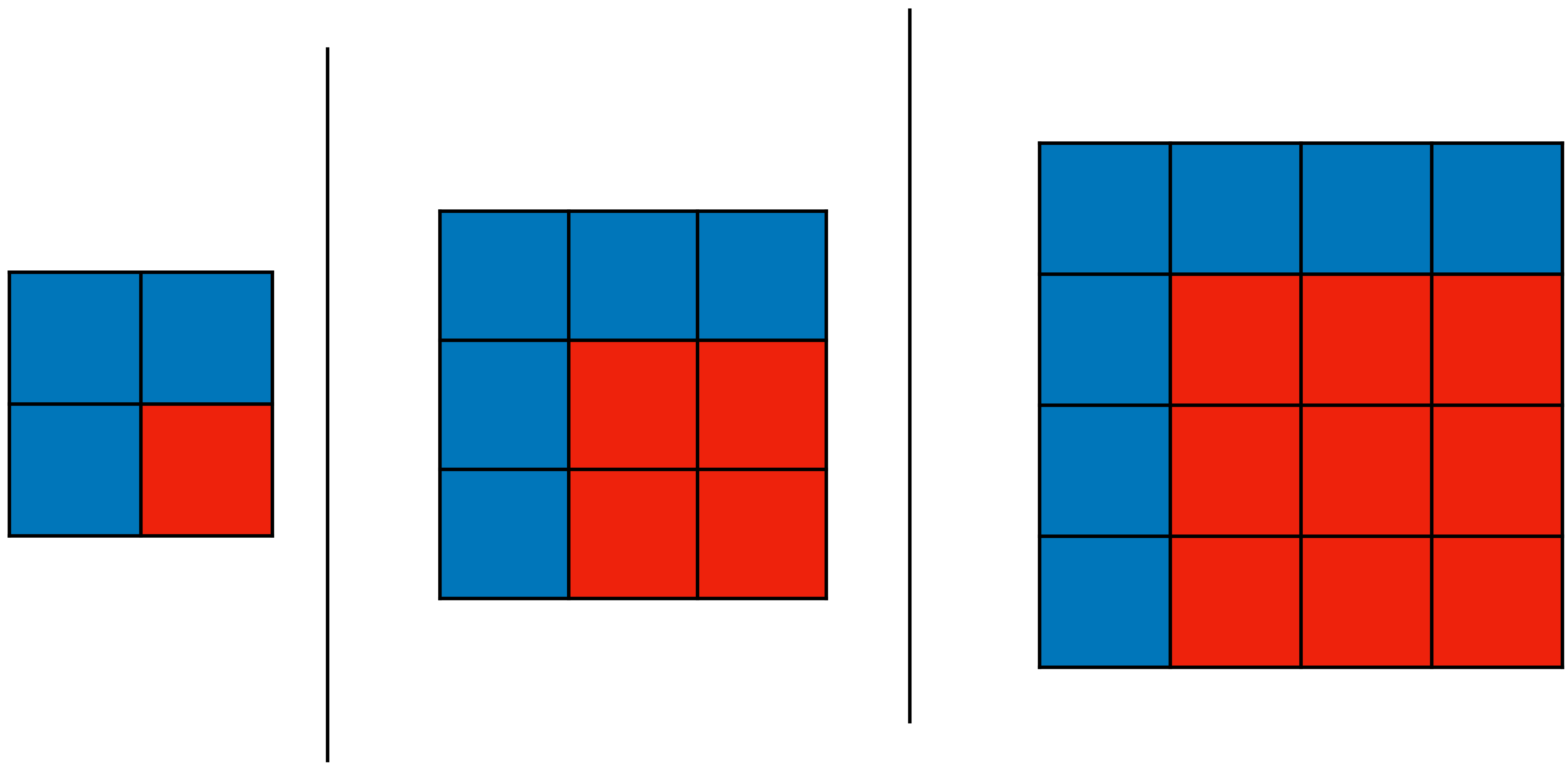
Exit Ticket

Write an algebraic equation that describes what happens to *any* number n when you apply the magic trick. That is, take n, then add 2, multiply by 2, subtract 2, divide by 2, and subtract the starting number n.
What is the result, in algebra?

What if you change all the 2s to 3s?
Write the algebraic equation for this too.

Day 6

Opener:	Visual Patterns
Task:	Pilgrim's Puzzle
Exit Ticket:	An algebraic description for what loops do, and why.



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Pilgrim's Puzzle

Topics: arithmetic operations, puzzles
Materials: pencil and paper, chalk for outside launch (optional), worksheet (optional)
Time: 1 - 2 lessons

How can the pilgrim reach the temple without paying a tax?

Why We Love Pilgrim's Puzzle

This original puzzle involves the interplay of all four arithmetic operations. Intuitively, it seems unsolvable, but an honest and satisfying solution exists, and students can make their way to it via a series of smaller breakthroughs in their understanding of what is possible in the structure of the problem.

The Launch

Launch option 1
show the video at https://www.youtube.com/watch?time_continue=1&v=6sBB-gRhfjE
Pause before the solution plays, and try a few sample pathways with students to make it clear how the problem works.

For example, if the pilgrim had already walked EE, as in the video, and then walked EESSSS, what would the tax be when she arrived? (128 silver)

If the pilgrim walked EE, then SSWWSSEEEE, what would her tax be? (56 silver)

If the pilgrim walked EE, then SWSESWSEEE, what would her tax be? (58 silver)

Once students understand the mechanic of the riddle, give them 5 - 10 minutes to try to solve the problem with a partner. Then step in to help with a simpler "starter" version of the riddle.

Simplified Pilgrim's Puzzle: when the Pilgrim comes to Duonia next time, she knows about the tax, and can plan her route from the NW corner of town. Also, she isn't on a pilgrimage this time, so can walk the same blocks more than once. How can she reach the temple without owing any silver in taxes?

Launch Option 2
Share the following story, along with the worksheet on the following page. Students can draw on the paper, or you can act out the story by having students walk through a chalk version of Duona outside the classroom, or with desks acting as the 16 blocks inside the classroom.

A pilgrim arrives in Duona, a sixteen-block town created by five streets running north-south that intersect with five streets running east-west. Like all pilgrims, she arrives in *the northwest corner of town, and needs to make her way to a shrine in the southeast corner. Unfortunately for the pilgrim, Duona imposes a tax system on visitors, charging 2 silver pieces for each block walked to the east, and doubling what you owe every time you walk south. To make the payment system fairer and encourage longer stays, they subtract 2 silver pieces for each block walked to the west, and halve what you owe when you walk a block north. The townsfolk keep track of your path, and you must pay in full on your arrival in the southeast corner. (Legend has it that certain savvy travelers have planned trips through the town and ended up receiving silver by following the tax rules.)*

The pilgrim has no money, and has no intention of leaving with any. How can she travel to the southeast corner of Duona without owing or receiving any silver?

The Work

Students should start by drawing and calculating different paths. Any reduction in the tax they're able to achieve should be celebrated!

After students have been working for a bit, they may be ready for a hint. There are two excellent hints to offer.

Hint #1. For the pilgrim to owe 0 when she arrives at the temple, what must she have owed just before that? What about before that?
The idea of this hint is that the pilgrim must have owed a *negative* tax before she owed 0. This sets up an intermediate step: how can we have the pilgrim owe a negative quantity?

Hint #2. What happens if the pilgrim circles a single block?
This is a powerful hint, and students should be able to observe that circling the block clockwise adds one to the tax owed, while circling counterclockwise subtracts one. This observation is powerful enough to lead very quickly to a solution to the simplified puzzle.

Prompts and Questions

- Try this path from the NW corner to the temple. What will it cost?
- Can you find another path that's less than that?
- (Hint #1) For the pilgrim to owe 0 when she arrives at the temple, what must she have owed just before that? What about before that?
- (Hint #2) What happens in the pilgrim circles a single block?
- Are you sure circling the block always subtracts one from the tax, no matter what you start at? How do you know?
- What happens if the pilgrim walks in circles around two blocks, or four blocks?
- I'm convinced you have a solution. Is it the quickest path with 0 tax?
- Can you solve the full version, where the pilgrim cannot walk the same block twice?

The Wrap
Bring students together and discuss their ideas on the problem. The central idea to highlight is what happens when the pilgrim circles a single block. Why does this work? If students know algebra, you could try using algebra to solve. (This could also segue into the Math Magic Trick as a sequel task in a future lesson.) Once students see how circling a block can reduce the tax owed by one silver, how can they use this to solve the simplified puzzle? The shortest solution with repetition of blocks is 12 blocks (SENWSSSSEEEE). The shortest solution without repetition is 14 blocks, and there are many of these. The video solution gives the only solution I know that begins with EE.

If students have solved the puzzle from the video, you can play the end of it for the class. If no one has solved the video puzzle, you can save it for a future time to prevent spoilers.

Extensions
- *The Greedy Pilgrim variation.*
  What if pilgrim wants to make as much money from the town without repeating her path? What route should she take? What is the most silver she can make?
- In Trionia, the tax system adds, subtracts, multiplies, and divides by 3 in each direction instead of 2. Is it possible to avoid the tax here?
- What about in Quadronia, where the number changes to 4 in each direction? Is there some theory that will tell you whether it is possible to find a pathway to the temple in any given tax system?
- See also the Math Magic Trick.

Tips for the Classroom
1. Acting out the riddle is a nice way to get everyone involved in the beginning. Students can walking around a grid of desks, or on a chalk outline outside the classroom.
2. If you show the video, make sure not to show the solution! If you launch without the video, you can show it later as a followup to the basic problem.
3. There are many answers to the simplified Pilgrim's puzzle.
4. The worksheet below uses "Duona" as the town name. This was updated to "Duonia" in the video.
5. Consider comparing this activity with the Math Magic Trick, which involves a deeper look into the mathematics of adding, multiplying, subtracting, and dividing by 2.

References.
See http://wordplay.blogs.nytimes.com/2013/08/26/pilgrim/ for the original.
See https://www.youtube.com/watch?v=6sBB-gRhfjE&t=8s for the TED-Ed riddle variation.

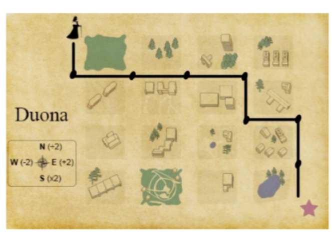
The Pilgrim's Puzzle

Image from the New York Times Numberplay column, by Gary Antonick

The Pilgrim is walking to the shrine at the SE corner of Duona. When she arrives, she'll have to pay a tax determined by her path. The tax she owes starts at 0 silver.

- When she travels a block East, she owes 2 more silver.
- When she travels a block West, she owes 2 less silver.
- When she travels a block South, the tax she owes doubles.
- When she travels a block North, the tax she owes is halved.

1. How much does the Pilgrim pay for each of the paths below?

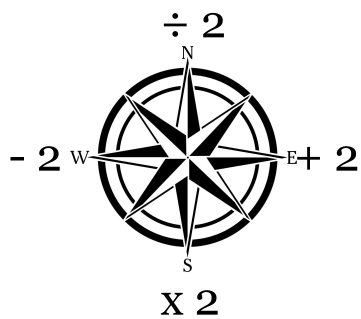


2. How can the pilgrim travel to the southeast corner of Duona without owing or receiving any silver?

3. How can the pilgrim make the trip without covering the same segment of street more than once?

4. (Greedy Pilgrim Variation) What's the most the Pilgrim can *earn* without repeating the same segment of street more than once?

Pilgrim’s Puzzle



Exit Ticket

Describe the effect of the pilgrim walking a clockwise or counterclockwise loop.

What happens? Why?

Day 7

Opener:	Visual Patterns
Task:	Step Numbers
Exit Ticket:	An algebraic description for 5-step numbers, and proof that they are multiples of 5.

Day 8

Opener:	Visual Patterns
Task:	Menu for Triangle Numbers
Wrap Up:	Give students longer today (i.e., 25 minutes) to discuss their algebraic descriptions for whichever of the problems they took on. Ideally, you could let students describe their solutions to all of the different problems they worked on, or at least the ones that everyone had a chance to work on. See the writeup for more.

Day 9

Opener:	Visual Patterns
Task:	Multiplication Table Sums
Exit Ticket:	Describe a formula for the sum of the numbers on an n by n multiplication table. Why does it work?

Day 10

Opener:	Visual Patterns
Task:	Dot Multiplication
Exit Ticket:	What is $4 \otimes 6$? What is $a \otimes b$?

Day 11

Opener:	Visual Patterns
Task:	Square Counting
Exit Ticket:	How many (non-tilted) squares are on a 10 by 10 checkerboard?
	How many (non-tilted) squares are on an *n* by *n* checkerboard?

Day 12

Opener:	Visual Patterns
Task:	Rectangle Counting
Exit Ticket:	How many (non-tilted) rectangles are on a 10 by 10 checkerboard?
	How many (non-tilted) rectangles are on an n by n checkerboard?

Day 13

Opener:	Visual Patterns
Task:	Triangle Counting
Exit Ticket:	Conjecture for the area of the *largest* triangle that could be drawn in an n by n grid, and why.

Day 14

Opener:	Visual Patterns
Task:	Tilted Squares
Exit Ticket:	The largest area square in an n by n grid has area n^2. What is the area of the *second* largest area square?

Day 15

Opener: Visual Patterns
Task: Finding Slopes
Exit Ticket: How many different slopes could a line have on a 4 by 4 grid? Explain.

Day 16

Opener: Visual Patterns
Task: Checker Challenge
Exit Ticket: Write an equation for the number of moves it takes to solve the checker challenge with 5 red and five black checkers.
Write an equation for n red and n black checkers.

Day 17

Opener:	Visual Patterns
Task:	Final task (to solve in pairs and write up): How many dots on the dominoes?