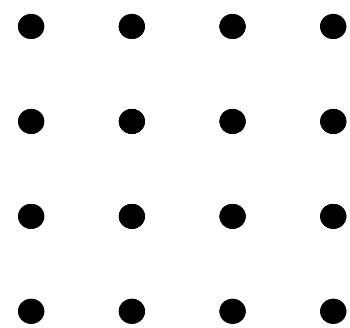
Square Counting

Topics: Counting and combinatorics, patterns, algebra
Materials: Pencil and paper, graph paper
Common Core: 5.OA.1, 5.OA.2, 5.OA.3, 6.EE.1, 6.EE.2, 6.EE.3, 7.EE.4, A-SSE.1, MP1, MP2, MP3, MP5, MP6, MP7, MP8

How many squares are on the grid?

Why We Love Square Counting

This twist on a classic problem emphasizes the critical work of conjectures, counterexamples, and the value of organizing data.



The Launch

Present a 4 by 4 grid of dots to the class, and pose the question of how many squares can be formed on the grid. One clarification may be necessary: for the squares to count, all their corners should be on the points of the grid.

You can open by claiming that there are exactly 9 squares that can be formed. Can students break that claim by finding a tenth square? Usually students will note that connecting the four outside points form an extra large square. Are there more after that? Let students work in pairs or small groups for a few minutes, then discuss answers as a class. As students give their answers, press them for how they know they have accounted for all the possible squares. There are two possibilities for where students might end up.

One is that everyone ends up agreeing that there are 14 squares: nine 1 by 1 squares, four 2 by 2 squares, and one 3 by 3 squares. In this case, go forward with the main question below:

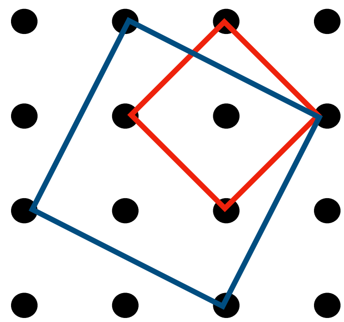
How many grid squares are there on different sized square grids?
In particular, how many squares on a 2 by 2 grid? A 3 by 3 grid? A 5 by 5 grid? What about a 10 by 10 grid?

A second possibility is that some student might notice that there are other squares lurking in the image! Specifically, there are “diagonal” squares, such as the ones below.

If students notice these squares (or even one of them), then they have found a fantastic counterexample that breaks open our earlier conceptions of the solution to this problem. In this case, you should disambiguate the questions into two versions:

- How many grid squares are there on different sized square grids if we only allow squares with horizontal/vertical sides?
- How many grid square are there on different sized square grids if we allow all of them, including the “diagonal” sides?

Students should work on the first question to start. If they can solve it completely, they can move on to the second question.



Examples of hard-to-find grid squares

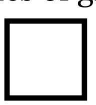
The Work

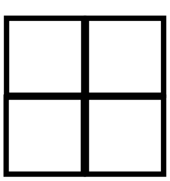
Students work in pairs or small groups to find the number of (horizontal/vertical) grid squares. Some students will be tempted to try large grids first. You can challenge them to with the question of how they know they counted *all* the squares. Students who start with smaller grids will usually find patterns more quickly.

There are two layers of organization that are helpful here, and which you can nudge students towards. One is the idea of keeping track of the data in a chart:

Grid Size	Number of Squares
2 by 2	1
3 by 3	5
4 by 4	14
5 by 5	30

The other organizational idea that turns out to be very useful is to track the different sizes of grid square on each grid.



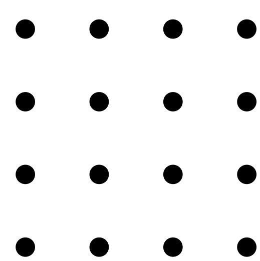


1 by 1 squares: 9

2 by 2 squares: 4

3 by 3 squares: 1

Total: 1 + 4 + 9 = 14



If students track their data using these two organizational structures, they are likely to notice one of several patterns in the data, and be able to use these patterns to extrapolate conjectures out to larger grids, e.g. a 10 by 10 grid, or, for students who know algebra, an *n* by *n* grid.

There are several important pushes you can give students even as they seem to have correct answers. First, are they just conjecturing the correct answer, or are they actually confirming that their guesses are correct? Until they have a direct count or some kind of argument confirming their guess is correct, those guesses should still have question marks next to them.

Grid Size	Number of Squares
2 by 2	1
3 by 3	5
4 by 4	14
5 by 5	30
6 by 6	55?

If students can come up with an argument that their method of predicting the number of squares actually works, then there are plenty of extending questions (see below).

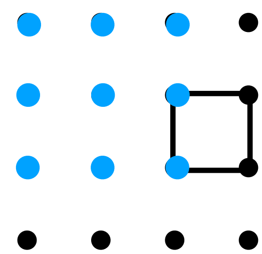
Prompts and Questions

- How do you know you've counted all the squares?
- How are you organizing your data?
- Let's try counting just one type of square at a time.
- Have you found any patterns?
- What's your guess for the number of squares on the next grid up?
- (If students know algebra) What's the solution for the number of squares on an *n* by *n* grid?
- What about a 50 by 50 grid? Is there a quick way to calculate this, or only a slow way?
- (If the question came up) What's the number of squares on the grid if we included slanted squares?

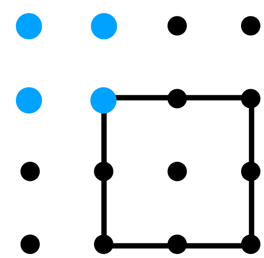
The Wrap

Bring the class together and let students share their solutions to the original problem only. The main argument we're looking for is how students can track the squares in an organized fashion.

One very nice way to approach this is to match the squares themselves to something simpler, such as a dot in their center or corner. If we mark each square with a dot in its upper left corner, for example, we'll see that there are nine 1 by 1 squares, since none can have their upper left corner on the bottom or right point of the grid.



For similar reasons, there are four 2 by 2 squares. And of course, only one 3 by 3 square, the largest square.



Expanding on this reasoning, the number of total squares on any size grid is always a sum of square numbers, i.e., $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots$

Students may come to this idea in other ways, of course, including inferring it from their data.

There is a quick way to find the sum of squares like this, but it probably isn't worth getting into unless students are familiar with other combinatorial problems, including triangle numbers and the fact that square numbers are the sums of consecutive odd numbers. If students are ready to look into finding the formula itself, a suggestive organization is to write the sum of square numbers as the sum of odd numbers arranged in triangles:

1 1 1 1 3 3 3 5 5 7	+	1 3 5 7 1 3 5 1 3 1	+	7 5 3 1 5 3 1 3 1 1

Students who are ready to tackle this question can use the image above to derive the formula for the sum of square numbers.

$$1^2 + 2^2 + 3^2 + 4^2 \ldots m^2 = (2m+1)(m+1)m/6$$

Tips for the Classroom

1. Keep students honest by differentiating between conjectures and actual confirmed counts of squares.
2. There are plenty of extension problems here, but don't feel a need to hurry into them, or even touch on them at all. The initial question is often rich enough to explore for a class period. Keep the extension problems in your back pocket only for students who need the extra challenge.
3. Once the lesson is done, a video animation might be nice for closing and discussion. One potential video can be found here: https://www.youtube.com/watch?v=aXbT37IlyZQ.