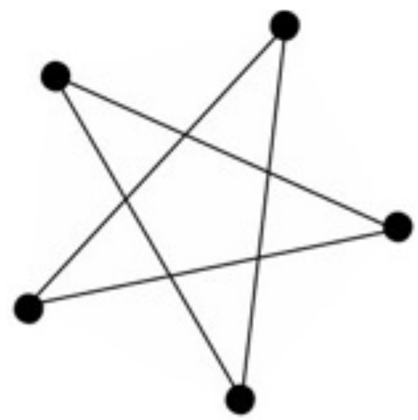
Constructing Star Polygons

The five-pointed star below was made by starting with 5 evenly spaced points and connecting every 2nd point. We say we've used the "connection rule 2."

It's called a *star polygon* when you hits **all the points** you started with in one continuous loop.



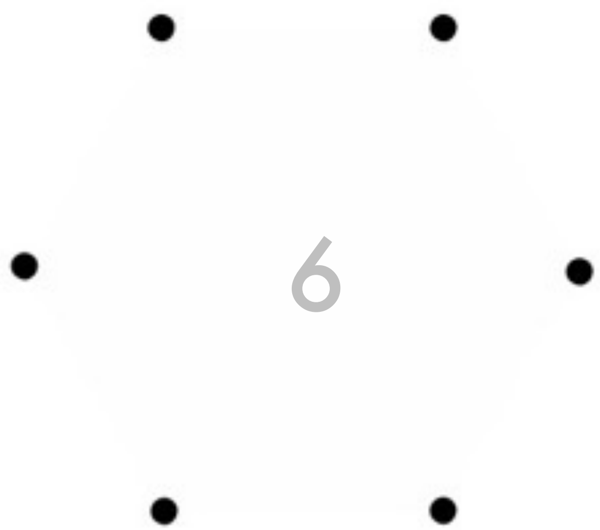
1. For each number of dots on the following sheets, what connection rules create star polygons (i.e., hit all the dots)? Which connection rules don't create star polygons?

2. Is there a way to know if a connection rule c will create a star polygon on n dots without trying it? Write a conjecture that explains what connection rules make star polygons, and defend or break your conjecture to another student at this station.

3. For $n = 29$ dots, which connection rules create star polygons?

4. For $n = 30$ dots, which connection rules create star polygons?

5. For any n, how many connection rules create star polygons?
   Hint: break this up into cases.
   What if n is prime? A product of distinct primes?
   A power of 2? A power of 3? A power of p prime?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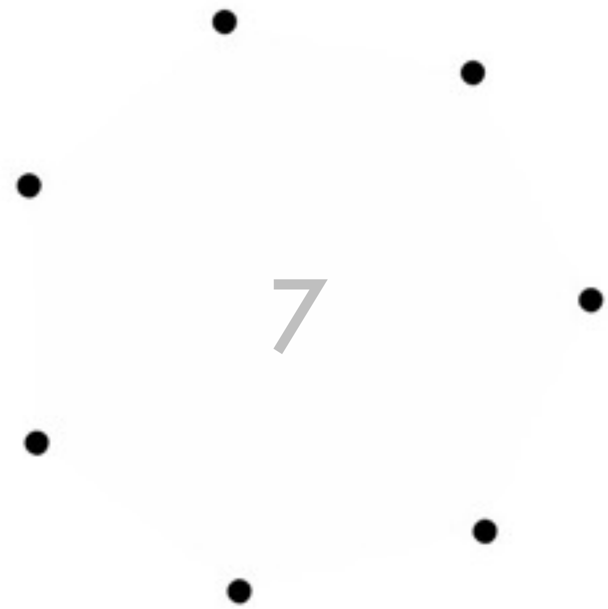
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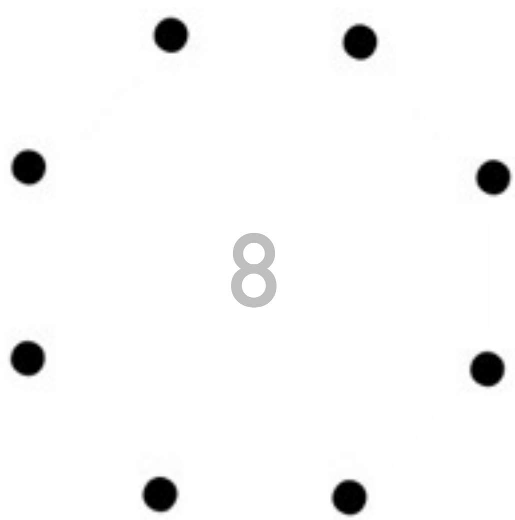
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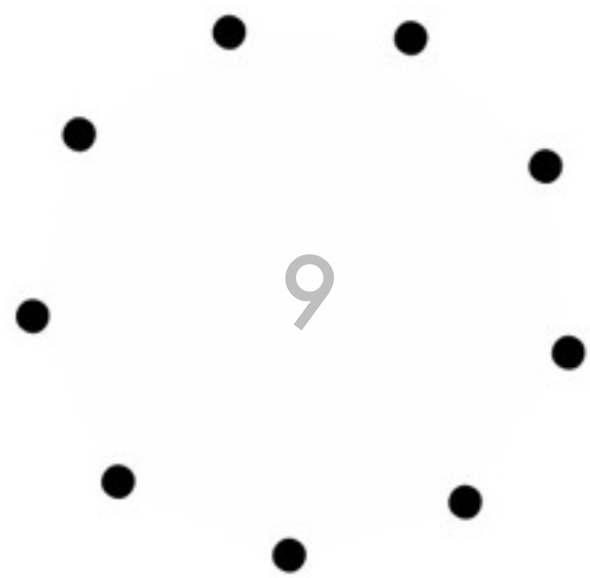
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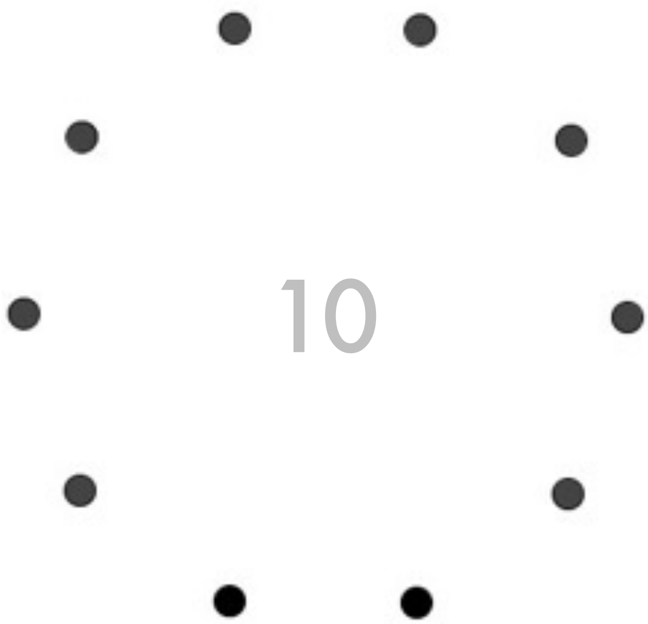
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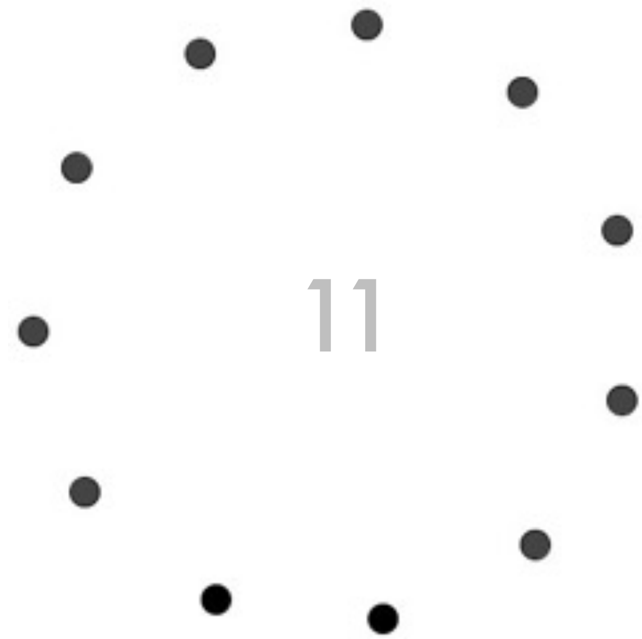
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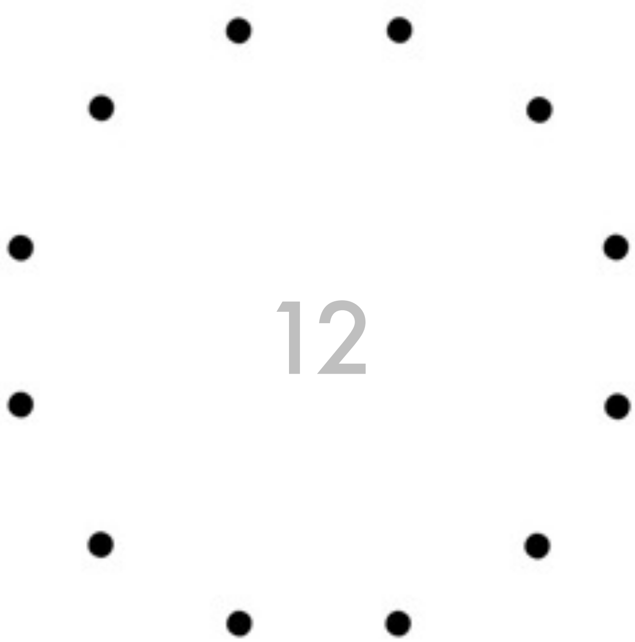
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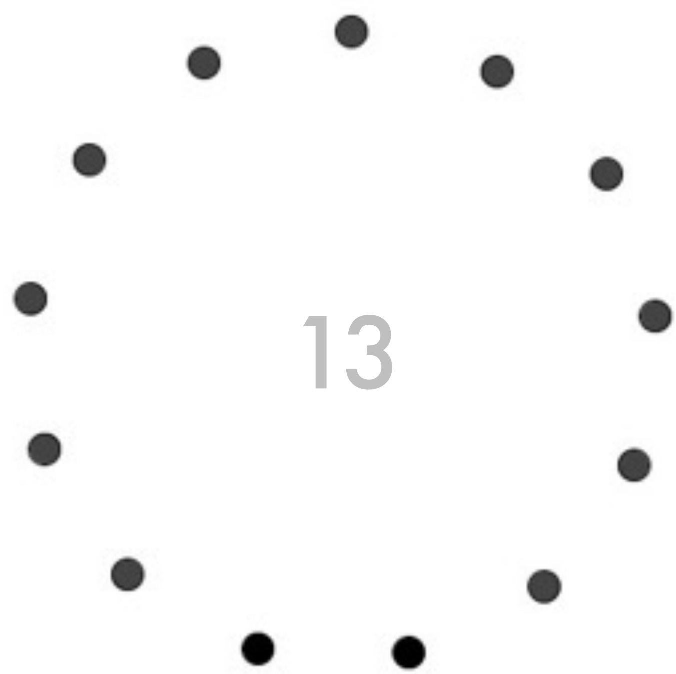
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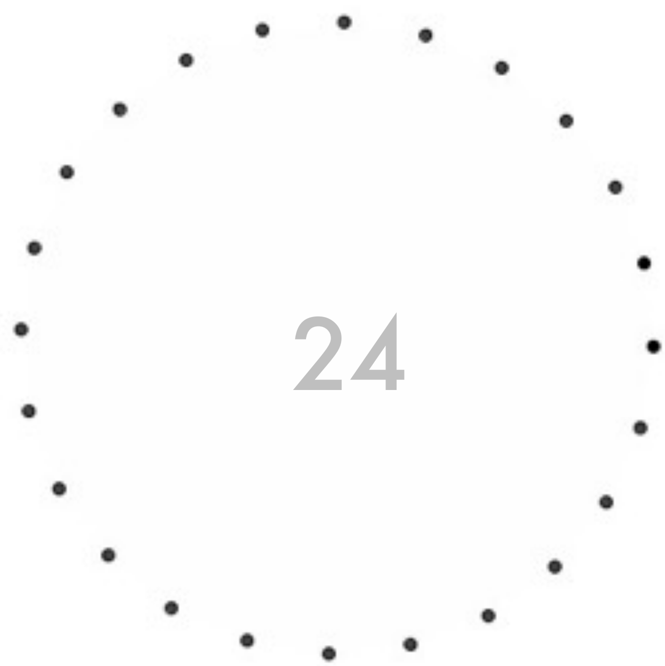
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Teachers Notes

The main thing is to make sure students understand how the connection rules work. Demonstrate as much as necessary. Once they understand the structure, they can explore on their own.

Crayons or colored pencils are helpful here.

Another very helpful suggestion is to get students tracking their data in a table:

Number of dots	Connection rules that lead to a star polygon	Connection rules that don't lead to star polygons
6	1, 5	2, 3, 4, 6
7	1, 2, 3, 4, 5, 6	7 (sidenote: should we call this a connection rule?)
8	1, 3, 5, 7	2, 4, 6, 8
9	1,2,4,5,7,8	3,6,9
10		
11		

Once they have that, they'll be able to make conjectures about what's going on. Then you can encourage them to disprove/defend each other's conjectures by doing more examples.
Conjecture. If D is even, and C is odd, then you get a star polygon.
Note: False. D = 10 and C = 5 gives a counterexample.

Conjecture. If the number of dots D is prime, any connection rule from 1 to D-1 will hit every dot, producing a star polygon.
Note: This is true.

Several pathways through this problem are possible, depending on students' insights.

Pathway 1: When do you get a square? A triangle? A five-pointed star?
Looking at particular shapes that emerge from dots and skip rules is a great way to connect to issues of ratio and proportionality. When, for example, do we end up drawing a five-pointed star? Certainly with five dots (D = 5) and connection rule 2 or 3 (C = 2,3). But also D = 10 and C = 4, 6. And D = 15 and C = 6, 9. Indeed, when the proportion D:C = 5:2 or 5:3, we get the 5-pointed star. Exploring this connection will likely allow students to delve into concrete and abstract understandings of ratio and proportion,

especially as they test their more and more general conjectures of this nature. They can continue to make conjectures until, ideally, they can articulate why this ratio perspective makes sense.

Pathway 2: A complete accounting for when star polygons occur and when they don't. When do we hit every point? If we apply our understanding from extension 1, we can see that when D and C have any common factor (greater than 1), there must be dots we miss. But what about when they don't have a factor in common? In this case, it turns out we do hit every point. Proving that this is indeed the case is an excellent project for older or more advanced students. For even more advanced students, counting the number of C that create star polygons for a given D is a very interesting and challenging project. (The answer is known as the *totient* or *Euler phi function*.)

There are also some nice variations/extensions to explore, for students who are ready.

Extension 1. What if C is larger than D? Trying connection rules greater than the number of dots leads to a potential exploration of modular arithmetic.

Extension 2. What are the angles of a star polygon? Exploring the angles leads to an entirely different, beautiful theory.