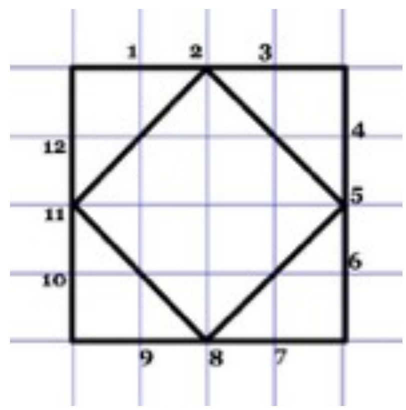
Quadrilateral Bisection 1

Topics: Area, Problem Solving, Geometry
Materials: Graph paper and pencils, Geoboard (optional), Launch Video (optional), Student Handout (optional)
Common Core: 6.G.1, 7.G.6, MP1, MP2, MP3, MP6, MP7

Will any of these quadrilaterals occupy half the square?

Why We Love Quadrilateral Bisection 1

This deceptively simple, seductive problem involves lots of practice finding the area of right triangles and quadrilaterals, and promotes an understanding of area that goes beyond formulas. With the right observations, a deeper insight makes the problem simpler, and gives rise to a new kind of numerical question. A great problem for differentiating across many skill levels.



The Launch

[See accompanying video, or see below]
Start with a square, shown on the graph paper on the right, draw three dots on each side, dividing each side into four equal segments. Connect the points 2, 5, 8, and 11.

Warmup. Prove that the area of the quadrilateral you get by connecting 2, 5, 8, 11 is half the area of the original square.

Give a minute or two, and let students prove that the inner quadrilateral (which is, in fact, a square) has area 1/2 the larger square. There are many ways to do this, including:

1. Finding the areas of the four right triangles in each corner of the large square and comparing them to the total area.
2. Cutting the inner square into right triangles.
3. "Folding" the outside triangle inward to see that they perfectly cover the inside square.

And so on.

Once students have finished the warmup problem, pose the question:

Will any other quadrilateral we make by connecting up the four sides take up exactly half the square?

Let students share their gut reactions, then try an example. There are a lot of other ways to connect these points, like 1, 4, 7, 10, or 1, 5, 8, 12. Pick one that *doesn't* take up half the square, like 1, 4, 8, 10, and find its area as a class. You can either demonstrate the whole process, or let students work on their own for a few minutes, then come together and discuss.

Once you've done the example, propose the following conjecture:

***Conjecture**. The quadrilateral 2, 5, 8, 11 is the only quadrilateral give by connecting all four sides of the square that takes up exactly half of the square.*

Let students go to work. Their job:

1. Find a counterexample to prove the conjecture is false. If they can find one, they can refine the conjecture to make it better.
   OR
2. Prove the conjecture is true.

The Work

While students may initially believe 2, 5, 8, 11 is the only quadrilateral that bisects the square, they will find counterexamples soon enough. The work progresses from there to try to discover if there is a way to see that a quadrilateral bisects the square without having to do all the calculations with the areas of the triangles, and to count how many of these quadrilaterals there are. Here is a progression of questions for students to tackle:

1. Is there a way to see that a quadrilateral bisects the square without having to do all the calculations with the areas of the triangles?
2. How many ways can you choose a point from each side and still get a quadrilateral that takes up half the original square?

More advanced questions include:

3. If you were to pick a point from each side at random, what are the odds that you would get a quadrilateral that takes up half the original square?
4. Let's generalize by increasing the number of points spaced along each side. If there were four points on each side instead of three, how many quadrilaterals would bisect the original square? What about five points? Or n points?
5. What if we did the same thing with a rectangle? Or a parallelogram? Would the count still be the same? If not, what would it be?
6. We could try this with a triangle as well. Start with an equilateral triangle, draw three equally placed points on each side, dividing them into quarters. If you connect the midpoints of each side, you get a new triangle with 1/4 the area of the starting triangle. How many of the other triangles will also have 1/4 the area of the starting triangle?

Prompts and Questions

- How are you organizing your work?

- Have you tried finding the area of 1, 4, 7, 10 yet [or some other specific example]? Try it.
- What do all the examples that have area 1/2 the original square have in common?
- You've disproved the conjecture. So what's your new conjecture for when a quadrilateral will bisect the original square?
- Do you think your new conjecture will hold? Or is there a counterexample?
- How can you prove it?

The Wrap

When most/all students have solved the first two questions in the progression of questions above, they can share how they solved the problem. (This could happen in one day, or take two or more.) Students should, at a minimum, be able to clearly find the area of any given quadrilateral.

It turns out that there are actually 45 different quadrilaterals that take up half the square, out of 81 total quadrilaterals. Here's a sketch of how you might count them:

Step 1: Prove that if two points of the quadrilateral are on the same vertical or horizontal line, then the quadrilateral bisects the square (draw the connecting line and look at the triangle above and below).

Step 2: (harder) Prove that any quadrilateral that doesn't have two points on the same vertical or horizontal line won't take up half the square. There are a lot of approaches to this. Use your judgment on how hard to press students on this point.

Step 3: Counting gives 27 quadrilaterals with two points on the same vertical, and 27 with two points on the same horizontal. But 9 have both, so the total is 27 + 27 - 9 = 45 quadrilaterals that bisect the square.

Amazingly, this gives a 45/81 = 5/9 chance of choosing one at random. More than half!

There are many generalizations (more points, different shapes—triangles, hexagons, cubes) that are fun to explore. We give just a few toward the end. Pushing deeper into the progression of questions can lead further down this rabbit hole, and into some interesting generalizations. Considering the square with n points on a side instead of three points on a side is an especially interesting generalization for an algebra class.

Tips for the Classroom

1. Graph paper is helpful. Rulers may be too, though not everyone requires them.
2. Question 1 will take up most of the time for most kids: it's a good, meaty problem. What you'll need to do is encourage them to keep trying out different quadrilaterals. They'll get the hang of it and start making conjectures once they amass enough evidence.
3. Check out http://wordplay.blogs.nytimes.com/2014/01/06/finkel/ for a backstory if you'd like to do a little more storytelling for this problem.